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Approximation of Invariant Measure for Damped Stochastic Nonlinear Schrödinger Equation via an Ergodic Numerical Scheme

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Abstract In order to inherit numerically the ergodicity of the damped stochastic nonlinear Schrödinger equation with additive noise, we propose a fully discrete scheme, whose spatial direction is based on spectral Galerkin method and temporal direction is based on a modification of the implicit Euler scheme. We not only prove the unique ergodicity of the numerical solutions of both spatial semi-discretization and full discretization, but also present error estimations on invariant measures, which gives order 2 in spatial direction and order $\frac{1}{2}$ in temporal direction under certain hypotheses.

Keywords Stochastic Schrödinger equation · Numerical scheme · Ergodicity · Invariant measure · Error estimation

Mathematics Subject Classifications (2010) 37M25 · 60-08 · 60H35 · 65C30

1 Introduction

The ergodicity of stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs) characterizes the longtime behavior of the solutions (see [5, 8, 14] and references therein), and it is natural to construct proper numerical schemes which could inherit the ergodicity. For ergodic SDEs with bounded or global Lipschitz coefficients,

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the ergodicity of several schemes were studied in [15]. It also gave an error estimation of invariant measures

$$e(\phi) = \left| \int \phi(y) d\mu(y) - \int \phi(y) d\tilde{\mu}(y) \right|$$

via the exponential decay property of the solution of Kolmogorov equation, where μ and $\tilde{\mu}$ denote the original invariant measure and the numerical one respectively. In the local Lipschitz case, the ergodicity is inherited by specially constructed implicit discretizations (see [14] and references therein). For SDEs, there are also various works related to the study of error $e(\phi)$ by assuming the ergodicity of the schemes (see [1] and references therein). For SPDEs, there have also been some significant results concentrating on invariant laws, e.g., [3] studied a semi-implicit Euler scheme in temporal direction with respect to parabolic type SPDEs with bounded nonlinearity and space-time white noise; [4] studied a full discretization for stochastic evolution equations with global Lipschitz nonlinearity and space-time white noise. Invariant laws of the approximations are, in general, possibly not unique. To our knowledge, there has been less work on constructing a fully discrete scheme to inherit the unique ergodicity of SPDEs up to now.

In this paper, we consider an initial-boundary problem of an ergodic one-dimensional damped stochastic nonlinear Schrödinger equation

$$\begin{cases} du = \big(\mathbf{i}\Delta u - \alpha u + \mathbf{i}\lambda |u|^2 u\big)dt + Q^{\frac{1}{2}} dW \\ u(t,0) = u(t,1) = 0, \ t \geq 0 \\ u(0,x) = u_0(x), \ x \in [0,1], \end{cases} \tag{1.1}$$

where $\alpha > 0$, $\lambda = \pm 1$ and the solution u is a complex valued ($\mathbb{C}$-valued) random field on a probability space $(\Omega, \mathcal{F}, P)$. The noise term involves a cylindrical Wiener process W and a symmetric, positive, trace class operator Q such that the noise is colored in space and white in time. The operator Q is supposed to commute with Laplacian Δ, and the noise has the following Karhunen-Loeve expansion

$$Q^{\frac{1}{2}} dW = \sum_{m=1}^{\infty} \sqrt{\eta_m} e_m(x) d\beta_m(t), \quad \eta_m \in \mathbb{R}^+ \ \text{ and } \ \eta := \sum_{m=1}^{\infty} \eta_m < \infty,$$

where $\{\beta_m(t)\}_{m \geq 1}$, associated to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, is a family of independent and identically distributed $\mathbb{C}$-valued Wiener processes and $\{e_m\}_{m \geq 1}$ is the eigenbasis of the Dirichlet Laplacian. This model has many applications in statistical physics and has been studied by many authors. For instance, it can describe the transmission of the signal along the fiber line with signal loss (see [11, 12] and references therein). The ergodicity for Eq. 1.1 with $\lambda = 1$ has been studied in [8] based on a coupling method, Foias-Prodi type estimates and a priori estimates for a modified Hamiltonian $\mathcal{H} = \frac{1}{2}\|\cdot\|_1^2 - \frac{1}{4}\|\cdot\|_{L^4}^4 + c_0\|\cdot\|_0^6$. The authors showed that (1.1) possesses a unique invariant measure μ assuming that the noise is non-degenerate in the low modes, i.e., $\eta_m > 0$, $m \leq N_*$ for some sufficiently large N_*. In the same procedure, one can also show the ergodicity for the cases $\lambda = 0$ and $\lambda = -1$ by setting $\mathcal{H} = \frac{1}{2}\|\cdot\|_1^2 - \frac{\lambda}{4}\|\cdot\|_{L^4}^4 + c_0\|\cdot\|_0^6$. Note that the damped term ($\alpha > 0$) is necessary for both linear and nonlinear Schrödinger equation to be ergodic.

Our work mainly focuses on the construction of a fully discrete and uniquely ergodic numerical scheme (i.e., whose numerical solution possesses a unique invariant measure). Moreover, the estimation of error between the original invariant measure and the numerical one is also considered based on the weak error of solutions.

In order to obtain a scheme whose noise remains in an explicit expression, we apply spectral Galerkin method in spatial direction to obtain a N-dimensional SDE

$$du_N = \left(\mathbf{i}\Delta u_N - \alpha u_N + \mathbf{i}\lambda \pi_N \left(|u_N|^2 u_N\right)\right)dt + \pi_N Q^{\frac{1}{2}} dW \tag{1.2}$$

with π_N being a projection operator. Here the spectral Galerkin method also ensures that the semigroup operator is the same as the one of Eq. 1.1, which simplifies the error estimate in spatial direction. We find a Lyapunov function by proving the uniform boundedness of u_N in L^2-norm. It ensures the existence of the invariant measure of Eq. 1.2. We show that the solution $u_N(t)$ is a strong Feller and irreducible process via the non-degeneracy of the noise term in Eq. 1.2. Hence, $u_N(t)$ possesses a unique invariant measure μ_N, which implies the ergodicity of $u_N(t)$. We would like to emphasize that the noise in the original equation do not need to be non-degenerate. Our method is also available under the same assumption in [8], that is $\eta_m > 0$, $m < N_*$ for some sufficiently large N_*. Here N and N_* need to satisfy the condition $N < N_*$ to ensure the non-degeneracy for the truncated noise and obtain the ergodicity for numerical solutions. The error between invariant measures μ_N and μ is transferred into the weak error of the solutions, which is required to be independent of time t. Different from conservative equations, the damped term in Eqs. 1.1 and 1.2 contributes to an exponential estimate on the difference between semigroup operators $S(t)$ and $S(t)\pi_N$, where $S(t)$ is generated by the linear operator $\mathbf{i}\Delta - \alpha$. Therefore, we achieve the time-independent weak error of solutions directly which, together with the ergodicity of u and u_N, deduces the error between invariant measures μ_N and μ.

For the temporal discretization of Eq. 1.2, we propose a new scheme

$$u_N^k - e^{-\alpha\tau} u_N^{k-1} = \left(\mathbf{i}\Delta u_N^k + \mathbf{i}\lambda \pi_N \left(\frac{|u_N^k|^2 + |e^{-\alpha\tau} u_N^{k-1}|^2}{2} u_N^k\right)\right)\tau + \pi_N Q^{\frac{1}{2}} \delta W_k, \tag{1.3}$$

which is a modification of the implicit Euler scheme. In order to analyze the effect of the time discretization, we investigate both the ergodicity of u_N^k and the weak error between u_N and u_N^k. The fully discrete scheme (1.3) is specially constructed to ensure the uniform boundedness of u_N^k in L^2-, $\dot{H}^1$- and $\dot{H}^2$-norms, which is essential to obtain the existance of the invariant measure as well as the time-independence of the weak error. Together with the Brouwer fixed point theorem and properties of homogeneous Markov chains, we prove that u_N^k is uniquely ergodic. For the weak error, it is usually analyzed in a finite time interval $[0, T]$ and depends on T (see e.g. [7, 9]). In our cases, however, the weak error between $u_N(T)$ and $u_N^M(T)$ is required to be independent of time T and step M. Thus, some technical estimates are given to obtain the exponential decay of the difference between non-global Lipschitz nonlinear terms and between $S(t)$ and S_τ. Based on the time-independency of the weak error of the solutions, we show that the error of invariant measures has at least the same order as the weak error of the solutions.

This paper is organized as follows. In Section 2, some notations and definitions about ergodicity are introduced. In Section 3, we apply spectral Galerkin method to Eq. 1.1 and prove the ergodicity of the spatial semi-discrete scheme. The time-independent weak error of the solutions, together with the error between invariant measures, is given. Section 4 is devoted to the proof of ergodicity of the fully discrete scheme. Moreover, we give the approximation error of invariant measure in temporal direction via the time-independent weak error. In Section 5, numerical experiments are given to verify the time independence of the weak error as well as the weak order in temporal direction for the linear case. The last section is the appendix of some proofs.

2 Preliminaries

In this section, we present some notations and the definition of ergodicity. Moreover, we introduce a sufficient condition for a stochastic process to be ergodic, which will be used in our proof on ergodicity of the numerical solution.

2.1 Notations

We set the linear operator $A := -\mathbf{i}\Delta + \alpha$, and the semigroup $S(t) := e^{-tA} = e^{t(\mathbf{i}\Delta-\alpha)}$ is generated by A. The mild solution of Eq. 1.1 exists globally and can be written as

$$u(t) = S(t)u_0 + \mathbf{i}\lambda \int_0^t S(t-s)|u(s)|^2 u(s)ds + \int_0^t S(t-s)Q^{\frac{1}{2}}dW(s).$$

It is obvious that $\{\lambda_n\}_{n\in\mathbb{N}} := \left\{\mathbf{i}(n\pi)^2 + \alpha\right\}_{n\in\mathbb{N}}$ is a sequence of eigenvalues of A with $1 \le |\lambda_n| \to +\infty$ and $\{e_n\}_{n\in\mathbb{N}} := \left\{\sqrt{2}\sin n\pi x\right\}_{n\in\mathbb{N}}$ is the associated eigenbasis of A with Dirichlet boundary condition. Denoting $L_0^2(0,1)$ as the space $L^2(0,1)$ with homogenous Dirichlet boundary condition, then $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L_0^2(0,1)$.

Definition 1 For all $s \in \mathbb{N}$, we define the normed linear space

$$\dot{H}^s := D(A^{\frac{s}{2}}) = \left\{u\middle| u = \sum_{n=1}^{\infty}(u,e_n)e_n \in L_0^2(0,1)\ s.t.\ \sum_{n=1}^{\infty}\left|(u,e_n)\right|^2|\lambda_n|^s < \infty\right\},$$

endowed with the s-norm

$$\|u\|_s := \left(\sum_{n=1}^{\infty}\left|\,(u,e_n)\,\right|^2|\lambda_n|^s\right)^{\frac{1}{2}},$$

where the inner product in the complex Hilbert space $L^2(0,1)$ is defined by

$$(u,v) = \int_0^1 u(x)\overline{v}(x)dx,\ \forall u,v \in L^2(0,1).$$

In particular, $\|u\|_0 = \|u\|_{L^2}, \forall u \in \dot{H}^0$.

In the sequel, we use notations $L^2 := L^2(0,1)$ and $H^s := H^s(0,1)$. It's easy to check that the above norms satisfy $\|u\|_r \le \|u\|_s (\forall\, 0 \le r \le s)$ and $\|u\|_s \cong \|u\|_{H^s} (s = 0,1,2)$ for any $u \in \dot{H}^s$.

The operator norm is defined as

$$\|B\|_{\mathcal{L}(\dot{H}^s,\dot{H}^r)} = \sup_{u\in\dot{H}^s}\frac{\|Bu\|_r}{\|u\|_s},\ \forall r,s \in \mathbb{N},$$

hence, for $0 \le r \le s$,

$$\|S(t)\|_{\mathcal{L}(\dot{H}^s,\dot{H}^r)} = \sup_{u\in\dot{H}^s}\frac{\left(\sum_{n=1}^{\infty}\left|\,\left(e^{t(\mathbf{i}\Delta-\alpha)}u,e_n\right)\right|^2|\lambda_n|^r\right)^{\frac{1}{2}}}{\|u\|_s} = \sup_{u\in\dot{H}^s}\frac{e^{-\alpha t}\|u\|_r}{\|u\|_s} \le e^{-\alpha t}.$$

We need $Q^{\frac{1}{2}}$ to be a Hilbert-Schmidt operator from L^2 to $\dot{H}^s$ with norm

$$\|Q^{\frac{1}{2}}\|^2_{\mathcal{HS}(L^2,\dot{H}^s)} := \sum_{m=1}^{\infty} \|Q^{\frac{1}{2}} e_m\|_s^2 = \sum_{m=1}^{\infty} |\lambda_m|^s \eta_m < \infty.$$

Assumptions on s will be given below.

2.2 Ergodicity

Let P_t be the Markov transition semigroup with an invariant measure μ and V be a Hilbert space. The Von Neumann theorem ensures that the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T P_t\phi(y)dt, \quad \phi \in L^2(V,\mu)$$

always exists in $L^2(V,\mu)$, where y denotes the initial value of the stochastic process.

Definition 2 (see e.g. [5]) If P_t has an invariant measure μ, and in addition it happens that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T P_t\phi(y)dt = \int_V \phi d\mu \quad in \quad L^2(V,\mu) \tag{2.1}$$

for all $\phi \in L^2(V,\mu)$. Then P_t is said to be ergodic.

Remark 1 In the following sections, we choose $P_t\phi(u_0) = E[\phi(u(t))|u(0) = u_0]$ for any deterministic initial value u_0, and take expectation of both sides of Eq. 2.1 to obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E[\phi(u)]dt = \int_V \phi d\mu \quad in \quad \mathbb{R}. \tag{2.2}$$

The sufficient conditions for a stochastic process to be ergodic are stated in the following theorem.

Theorem 2.1 *(see e.g. [5]) Let $F : V \to [0,\infty]$ be a Borel function (Lyapunov function) whose level sets*

$$L_a := \{x \in V : F(x) \le a\}$$

are compact for any $a > 0$. Assume that there exists $y \in V$ and $C(y) > 0$ such that

$$E\Big[F\big(u(t;y)\big)\Big] \le C(y) \quad for \ \ all \ \ t \in \mathbb{R}^+,$$

where $u(t;y)$ denotes a stochastic process whose start point is y. Then u has at least one invariant measure.

If in addition the associated semigroup P_t is strong Feller and irreducible, then u possesses a unique invariant measure. Thus, u is ergodic.

For Eq. 1.1, it is ergodic with a unique invariant measure.

Theorem 2.2 *(see [8]) There exists a unique stationary probability measure μ of $\{P_t\}_{t\in\mathbb{R}^+}$ on $H_0^1(0,1)$. Moreover, for any $p \in \mathbb{N}\backslash\{0\}$, μ satisfies*

$$\int_{H_0^1(0,1)} \|u\|_1^{2p} d\mu < \infty.$$

3 Spatial Semi-discretization

We apply spectral Galerkin method to problem (1.1) to get a spatial semi-discrete scheme which is a finite-dimensional SDE. We show that the solution u_N of Eq. 3.1 possesses a unique invariant measure μ_N, which leads to the ergodicity of u_N. Furthermore, we prove that the weak error of the spatial semi-discrete scheme does not depend on the time interval, which implies that μ_N converges to μ in at least the same rate.

3.1 Spectral Galerkin Method

The finite-dimensional spectral space is defined as

$$V_N := span\{e_m\}_{m=1}^N.$$

Let $\pi_N : \dot{H}^0 \to V_N$ be a projection operator, which is defined as

$$\pi_N u = \sum_{m=1}^N (u, e_m)e_m, \ \forall\, u = \sum_{m=1}^\infty (u, e_m)e_m \in \dot{H}^0.$$

We use u_N as an approximation to the original solution u, and the spatial semi-discrete scheme is expressed as

$$\begin{cases} du_N = \Big(\mathbf{i}\Delta u_N - \alpha u_N + \mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right)\Big)dt + \pi_N Q^{\frac{1}{2}}dW \\ u_N(0,x) = \pi_N u_0(x), \end{cases} \tag{3.1}$$

where $\pi_N Q^{\frac{1}{2}}dW = \sum_{m=1}^N \sqrt{\eta_m}e_m(x)d\beta_m(t)$, and the projection operator π_N is bounded

$$\|\pi_N\|_{\mathcal{L}(\dot{H}^s,L^2)} \le 1, \quad \forall s \in \mathbb{N}.$$

3.2 Ergodicity of Spatial Semi-discrete Scheme

Theorem 3.1 *Let $u_N(t,x)$ be the solution of Eq. 3.1, then u_N possesses a unique invariant measure, denoted by μ_N. Thus, u_N is ergodic.*

Proof Following from Theorem 2.1, we need to show three properties of u_N,"strong Feller", "irreducibility" and "Lyapunov condition", in order to show the ergodicity of u_N. Thus the proof is divided into three parts as follows.

Part 1. Strong Feller. We transform (3.1) into its equivalent finite-dimensional SDE form. Denote $a_m(t) = \big(u_N(t,x), e_m(x)\big)$ and we have

$$u_N(t,x) = \sum_{m=1}^N a_m(t)e_m(x).$$

Applying the Itô's formula to $a_m(t)$ leads to

$$da_m(t) = \Big[-\lambda_m a_m(t) + \Big(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_m\Big)\Big]dt + \sqrt{\eta_m}d\beta_m(t), \quad 1 \le m \le N. \tag{3.2}$$

We decompose the above equation into its real and imaginary parts by denoting $a_m = a_m^1 + \mathbf{i}a_m^2$, $\lambda_m = \lambda_m^1 + \mathbf{i}\lambda_m^2$ and $\beta_m = \beta_m^1 + \mathbf{i}\beta_m^2$, where $\{\beta_m^i\}_{1\le m\le N, i=1,2}$ is a family of independent $\mathbb{R}$-valued Wiener processes and the superscripts 1 and 2 mean the real and imaginary parts of a complex number, respectively, and obtain

$$\begin{cases} da_m^1 = \Big[-\lambda_m^1 a_m^1 + \lambda_m^2 a_m^2 + Re\left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_m\right)\Big]dt + \sqrt{\eta_m}d\beta_m^1(t), \\ da_m^2 = \Big[-\lambda_m^2 a_m^1 - \lambda_m^1 a_m^2 + Im\left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_m\right)\Big]dt + \sqrt{\eta_m}d\beta_m^2(t). \end{cases}$$

With notations $X(t) = (a_1^1(t), a_1^2(t), \cdots, a_N^1(t), a_N^2(t))^T$, $\beta = (\beta_1^1, \beta_1^2, \cdots, \beta_N^1, \beta_N^2)^T \in \mathbb{R}^{2N}$, $F = diag\{\Lambda_1, \cdots, \Lambda_N\}$,

$$\Lambda_i = \begin{pmatrix} -\lambda_i^1 & \lambda_i^2 \\ -\lambda_i^2 & -\lambda_i^1 \end{pmatrix}, \quad G(X(t)) = \begin{pmatrix} Re\ \left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_1\right) \\ Im\ \left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_1\right) \\ \vdots \\ Re\ \left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_N\right) \\ Im\ \left(\mathbf{i}\lambda\pi_N\left(|u_N|^2 u_N\right), e_N\right) \end{pmatrix}$$

and

$$Z = \begin{pmatrix} \sqrt{\eta_1} & & & & \\ & \sqrt{\eta_1} & & & \\ & & \ddots & & \\ & & & \sqrt{\eta_N} & \\ & & & & \sqrt{\eta_N} \end{pmatrix} := (Z_1^1, Z_1^2 \cdots, Z_N^1, Z_N^2),$$

we get an equivalent form of Eq. 3.1

$$dX(t) = \Big[FX(t) + G\big(X(t)\big)\Big]dt + \sum_{m=1}^{N}\sum_{i=1}^{2} Z_m^i d\beta_m^i := Y\left(X(t)\right)dt + \sum_{m=1}^{N}\sum_{i=1}^{2} Z_m^i d\beta_m^i.$$

It is obvious that

$$span\{Z_1^1, Z_1^2, \cdots, Z_N^1, Z_N^2\} = \mathbb{R}^{2N},$$

which means the Hörmander's condition holds. According to the Hörmander theorem [13], $X(t)$ is a strong Feller process.

Part 2. Irreducibility. By using the same notations as above, we have

$$dX = Y(X)dt + Zd\beta, \tag{3.3}$$

with $X = X(t) \in \mathbb{R}^{2N}$, $X(0) = y$ and Z being invertible. Using a similar technique as [14], we consider the associated control problem

$$d\overline{X} = Y(\overline{X})dt + ZdU, \tag{3.4}$$

with $\overline{X} = \overline{X}(t)$ and a smooth control function $U \in C^1(0, T)$. For any fixed $T > 0$, $y \in \mathbb{R}^{2N}$ and $y^+ \in \mathbb{R}^{2N}$, using polynomial interpolation, we derive a continuous function $\big(\overline{X}(t),\ t \in [0, T]\big)$ such that $\overline{X}(0) = y$ and $\overline{X}(T) = y^+$. Hence,

$$dU = Z^{-1}\big(d\overline{X} - Y(\overline{X})dt\big),$$

and we get the control function U such that (3.4) is satisfied with $\overline{X}(0) = y$, $\overline{X}(T) = y^+$ and $U(0) = 0$. We subtract the resulting Eqs. 3.3 and 3.4, and achieve

$$X(t) - \overline{X}(t) = \int_0^t Y(X(s)) - Y(\overline{X}(s))ds + Z(\beta(t) - U(t)), \quad t \in [0, T].$$

According to the properties of Brownian motion,

$$P\left(\sup_{0 \le t \le T} \left|\beta(t) - U(t)\right| \le \epsilon\right) > 0, \quad \forall \epsilon > 0.$$

Note that Y is locally Lipschitz because of its continuous differentiability, and the ranges of $X(t)$ and $\overline{X}(t)$ ($t \in [0, T]$) are both compact sets. Thus, it holds

$$P\left(\left|X(t) - \overline{X}(t)\right| \le \int_0^t C_1\left|X(s) - \overline{X}(s)\right|ds + C_2\epsilon, \quad \forall\, t \in [0, T]\right) > 0, \quad \forall\, \epsilon > 0$$

with C_1 and C_2 are positive constants independent of ϵ. Then the Grönwall's inequality yields

$$P\left(\left|X(t) - \overline{X}(t)\right| \le C_2(1 + e^{C_1 t})\epsilon, \quad \forall\, t \in [0, T]\right) > 0, \quad \forall\, \epsilon > 0.$$

For any $\delta > 0$, choosing $t = T$ and $\epsilon = \delta / C_2(1 + e^{C_1 T}) > 0$, we finally obtain

$$P\left(|X(T) - y^+| < \delta\right) > 0.$$

In other words, $X(T)$ hits $B(y^+, \delta)$ with positive probability. The irreducibility has been proved.

The above two conditions ensure the uniqueness of the invariant measure of $X(t)$. It suffices to show the existence of invariant measures in the following.

Part 3. Lyapunov condition. A useful tool for proving existence of invariant measures is provided by Lyapunov functions, which is introduced in Theorem 2.1. Itô's formula applied to $\|u_N(t)\|_0^2$ implies that

$$d\|u_N(t)\|_0^2 = -2\alpha\|u_N(t)\|_0^2 dt + 2Re\int_0^1 \overline{u}_N(t)\pi_N Q^{\frac{1}{2}} dx dW(t) + 2\sum_{m=1}^N \eta_m dt, \tag{3.5}$$

where we have used the fact that

$$Re\left[\mathbf{i}\lambda\int_0^1 \pi_N(|u_N|^2 u_N)\overline{u}_N dx\right] = Re\left[\mathbf{i}\lambda\int_0^1 \left(|u_N|^4 - (Id - \pi_N)(|u_N|^2 u_N)\overline{u}_N\right) dx\right]$$
$$= -\lambda Im\left((Id - \pi_N)(|u_N|^2 u_N), u_N\right) = 0.$$

Taking expectation on both sides of Eq. 3.5, we get

$$\frac{d}{dt}E\|u_N(t)\|_0^2 = -2\alpha E\|u_N(t)\|_0^2 + C_N,$$

where $C_N = 2\sum_{m=1}^N \eta_m \le 2\eta$. It is solved as

$$E\|u_N(t)\|_0^2 = e^{-2\alpha t}\left(\int_0^t C_N e^{2\alpha s}ds + E\|u_N(0)\|_0^2\right) \le e^{-2\alpha t}E\|u_N(0)\|_0^2 + C, \ \forall t > 0.$$

On the other hand,

$$\|u_N(t)\|_0^2 = \int_0^1 \left|\sum_{m=1}^N a_m(t)e_m(x)\right|^2 dx = \|X(t)\|_{l^2(\mathbb{R}^{2N})}^2.$$

Define $F = \|\cdot\|_{l^2(\mathbb{R}^{2N})} : \mathbb{R}^{2N} \to [0, +\infty]$. The level sets of F are tight by Heine-Borel theorem. Therefore, $X(t)$ is ergodic. We mention that the ergodicity of $X(t)$ is equivalent to the existence of a random variable $\xi = (\xi_1^1, \xi_1^2, \cdots, \xi_N^1, \xi_N^2)$ such that

$$\lim_{t\to\infty} X(t) = \xi, \text{ i.e., } \lim_{t\to\infty} a_m^i(t) = \xi_m^i, \ \forall\, m = 1, \cdots, N, \ i = 1, 2.$$

It leads to

$$\lim_{t\to\infty} u_N(t) = \sum_{m=1}^{N} \left(\xi_m^1 + \mathbf{i}\xi_m^2\right) e_m,$$

which shows the ergodicity of $u_N(t)$. □

According to the proof of Lyapunov condition, we have the following uniform boundedness for 0-norm. Moreover, 1-norm and 2-norm are also uniformly bounded, which is stated in the following proposition. Its proof is given in Appendix "The Proof of Proposition 3.1" for readers' convenience. In sequel, all the constants C are independent of the end point T of time interval and may be different from line to line.

Proposition 3.1 *Assume that* $u_0 \in \dot{H}^1$, $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^1)} < \infty$ *and* $p \geq 1$. *There exists positive constants* c_0 *and* $C = C(\alpha, p, u_0, c_0, Q)$, *such that for any* $t > 0$,

$$\begin{aligned} &i) \quad E\|u_N(t)\|_0^{2p} \leq e^{-2\alpha p t} E\|u_N(0)\|_0^{2p} + C \leq C, \\ &ii) \quad E\mathcal{H}(u_N(t))^p \leq e^{-\alpha p t} E\mathcal{H}(u_N(0))^p + C \leq C, \end{aligned}$$

where $\mathcal{H}(u_N(t)) = \frac{1}{2}\|\nabla u_N(t)\|_0^2 - \frac{\lambda}{4}\|u_N(t)\|_{L^4}^4 + c_0\|u_N(t)\|_0^6$. *In addition, if we assume further* $u_0 \in \dot{H}^2$ *and* $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$, *we also have*

$$iii) \ E\|u_N(t)\|_2^2 \leq C.$$

Remark 2 The uniform boundedness of the original solution u can also be obtained in the same procedure as Proposition 3.1 or [8]. As the $\dot{H}^2$-regularity for both the original solution and numerical solutions are essential to obtain the time-independent weak error, we need the assumption $u_0 \in \dot{H}^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$ in the error analysis.

3.3 Weak Error between Solutions u and u_N

Weak convergence is established for the spatial semi-discretization (3.1) in this section utilizing a transformation of $u_N(t)$ and the corresponding Kolmogorov equation.

Theorem 3.2 *Assume that* $u_0 \in \dot{H}^2$ *and* $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$. *For any* $\phi \in C_b^2(L^2)$, *there exists a constant* $C = C(u_0, \phi, Q)$ *independent of T, such that for any* $T > 0$,

$$\left| E\Big[\phi\big(u_N(T)\big)\Big] - E\Big[\phi\big(u(T)\big)\Big] \right| \leq CN^{-2}.$$

Before the proof of Theorem 3.2, we give a useful lemma.

Lemma 1 *Assume that $S(t)$ and π_N are defined as before. We have the following estimation*

$$\|S(t) - S(t)\pi_N\|_{\mathcal{L}(\dot{H}^s, L^2)} \leq Ce^{-\alpha t}N^{-s}.$$

Proof For any $u \in \dot{H}^s$, we have

$$\|S(t)u - S(t)\pi_N u\|_0 = e^{-\alpha t}\|u - \pi_N u\|_0 = e^{-\alpha t}\left(\sum_{n=N+1}^{\infty}|(u, e_n)|^2\right)^{\frac{1}{2}}$$

$$\leq e^{-\alpha t}|\lambda_N|^{-\frac{s}{2}}\left(\sum_{n=N+1}^{\infty}|\lambda_n|^s|(u, e_n)|^2\right)^{\frac{1}{2}} \leq Ce^{-\alpha t}N^{-s}\|u\|_s.$$

□

Proof of Theorem 3.2 We split the proof in three steps.

Step 1. Calculation of $E[\phi(u(T))]$.

To eliminate the unbounded Laplacian operator, we consider the modified process $Y(t) = S(T-t)u(t)$, $t \in [0, T]$, which is the solution of the following SPDE

$$\begin{aligned} dY(t) &= \mathbf{i}\lambda S(T-t)\Big[|S(t-T)Y(t)|^2 S(t-T)Y(t)\Big]dt + S(T-t)Q^{\frac{1}{2}}dW \\ &:= H(Y(t))dt + S(T-t)Q^{\frac{1}{2}}dW. \end{aligned}$$

Denote $v(T-t, y) := E[\phi(Y(T))|Y(t) = y]$ and it follows easily

$$\frac{\partial v(T-t, y)}{\partial t} = -\Big(Dv(T-t, y), H(y)\Big) - \frac{1}{2}Tr\Big[(S(T-t)Q^{\frac{1}{2}})^* D^2 v(T-t, y)S(T-t)Q^{\frac{1}{2}}\Big].$$

Note that the mild solution of u has the expression $u(T) = S(T-t)u(t) + \mathbf{i}\lambda\int_t^T S(T-s)|u|^2 u ds + \int_t^T S(T-s)Q^{\frac{1}{2}}dW$. Thus, we have

$$\begin{aligned} v(T-t, y) &= E[\phi(Y(T))|Y(t) = y] = E[\phi(u(T))|u(t) = S(t-T)y] \\ &= E\left[\phi\left(y + \mathbf{i}\lambda\int_t^T S(T-s)|u(s)|^2 u(s)ds + \int_t^T S(T-s)Q^{\frac{1}{2}}dW\right)\right]. \end{aligned}$$

For any $h \in L^2$, similar to [7] (Lemma 5.13), we have

$$(Dv(T-t, y), h) = E\left[\left(D\phi\left(y + \mathbf{i}\lambda\int_t^T S(T-s)|u(s)|^2 u(s)ds + \int_t^T S(T-s)Q^{\frac{1}{2}}dW\right), \chi^h(t)\right)\right]$$

with $\chi^h(t) = h + \mathbf{i}\lambda\int_t^T S(T-s)\left(2|u(s)|^2\chi^h(s) + u^2(s)\overline{\chi^h(s)}\right)ds$. It's easy to obtain that

$$\|\chi^h(t)\|_0 \leq \|h\|_0 + C\int_t^T e^{-\alpha(T-s)}\|u(s)\|_1^2\|\chi^h(s)\|_0 ds. \tag{3.6}$$

To show the uniform boundedness of $E\|\chi^h(t)\|_0$, we define a family of subsets

$$K_m := \left\{\omega \in \Omega \,\Big|\, \sup_{t\leq s\leq T}\|u(s)\|_1 > m(T+1-t)^{\frac{1}{2}}\right\}, \quad m \in \mathbb{N}$$

for any $t \leq T$. We claim that $E\left(\sup_{t\leq s\leq T}\|u(s)\|_1^2\right) \leq C + C(T-t)$. In fact, we can deduce

$$d\mathcal{H}(u(t)) \leq -\frac{3}{2}\alpha\mathcal{H}(u(t)) + Cdt + dM_*(t)$$

similar to Proposition 3.1 or [8], which implies

$$\mathcal{H}(u(s)) \leq e^{-\frac{3}{2}\alpha(s-t)}\mathcal{H}(u(t)) + \int_t^s Ce^{-\frac{3}{2}\alpha(s-r)}dr + \int_t^s e^{-\frac{3}{2}\alpha(s-r)}dM_*(r)$$

with $dM_* := 6c_0\|u\|_0^4 Re\left(u, Q^{\frac{1}{2}}dW\right) - Re\left(\Delta u + \lambda|u|^2u, Q^{\frac{1}{2}}dW\right)$ and $E\mathcal{H}(u(t)) \leq C$. Taking supremum and expectation, we get

$$\begin{aligned} E\left[\sup_{t\leq s\leq T}\mathcal{H}(u(s))\right] &\leq E\mathcal{H}(u(t)) + C(T-t) + E\left[\sup_{t\leq s\leq T}\int_t^s e^{-\frac{3}{2}\alpha(s-r)}dM_*(r)\right] \\ &\leq C + C(T-t), \end{aligned}$$

where in the last step we have used the Doob's inequality for convolution integrals (see [16], Theorem 2). This complete the proof of the claim. Then the Chebyshev's inequality (see e.g. [10]) yields that

$$P(K_m) \leq \frac{E\left(\sup_{t\leq s\leq T}\|u(s)\|_1^2\right)}{m^2(T+1-t)} \leq \frac{C+C(T-t)}{m^2(T+1-t)} \leq \frac{C}{m^2}, \quad \forall\, t \leq T.$$

As $\sum_{m=1}^{\infty} P(K_m) \leq \sum_{m=1}^{\infty}\frac{C}{m^2} < \infty$, we get $P(\cap_{n=1}^{\infty}\cup_{m=n}^{\infty}K_m) = 0$ based on the Borel-Cantelli Lemma (see e.g. [10]). It implies that there exists a constant $M_* \in \mathbb{N}$, for any $m \geq M_*$, $\|u(t)\|_1 \leq \sup_{t\leq s\leq T}\|u(s)\|_1 \leq m(T+1-t)^{\frac{1}{2}}$ almost surely. Then the backward Grönwall's inequality applied to Eq. 3.6 yields $E\|\chi^h(t)\|_0 \leq C\|h\|_0$ thanks to the exponential decay factor, and it holds

$$|(Dv(T-t,y),h)| \leq \|\phi\|_{C_b^1}E\|\chi^h(t)\|_0 \leq C\|\phi\|_{C_b^1}\|h\|_0. \tag{3.7}$$

Similarly, we also have

$$\left|\left(\left(D^2v(T-t,y),h\right),h\right)\right| \leq C\|\phi\|_{C_b^2}\|h\|_0^2. \tag{3.8}$$

The Itô's formula gives that

$$\begin{aligned} dv(T-t,Y(t)) &= \frac{\partial v}{\partial t}(T-t,Y(t))dt + \Big(Dv\left(T-t,Y(t)\right), H\left(Y(t)\right)dt \\ &\quad + S(T-t)Q^{\frac{1}{2}}dW(t)\Big) \\ &\quad + \frac{1}{2}Tr\left[(S(T-t)Q^{\frac{1}{2}})^*D^2v\left(T-t,Y(t)\right)S(T-t)Q^{\frac{1}{2}}\right]dt \\ &= \left(Dv(T-t,Y(t)), S(T-t)Q^{\frac{1}{2}}dW(t)\right). \end{aligned}$$

Therefore,

$$v(0,Y(T)) = v(T,Y(0)) + \int_0^T\left(Dv(T-s,Y(s)), S(T-s)Q^{\frac{1}{2}}dW(s)\right). \tag{3.9}$$

Noticing that $Y(0) = S(T)u_0$ and $Y(T) = u(T)$, we recall $v(T-t, y) = E[\phi(Y(T))|Y(t) = y]$ to derive

$$v(0, Y(T)) = E\left[\phi(u(T))|Y(T) = u(T)\right]$$

and

$$\begin{aligned} v(T, Y(0)) &= E\left[\phi(Y(T))|Y(0) = S(T)u_0\right] \\ &= E\left[\phi\Big(S(T)u_0 + \int_0^T H(Y(t))dt + S(T-t)Q^{\frac{1}{2}}dW(t)\Big)\Big|Y(0) = S(T)u_0\right]. \end{aligned}$$

Take expectation of both sides of Eq. 3.9 and we have

$$E[\phi(u(T))] = E\left[\phi\Big(S(T)u_0 + \int_0^T H(Y(t))dt + S(T-t)Q^{\frac{1}{2}}dW(t)\Big)\right]. \tag{3.10}$$

Step 2. Calculation of $E\left[\phi(u_N(T))\right]$.

The mild solution of Eq. 3.1 is

$$u_N(t) = S(t)\pi_N u_0 + \mathbf{i}\lambda \int_0^t S(t-s)\pi_N \left(|u_N(s)|^2 u_N(s)\right) ds + \int_0^t S(t-s)\pi_N Q^{\frac{1}{2}} dW(s).$$

Using similar argument as above, we consider the following stochastic process:

$$Y_N(t) = S(T-t)u_N(t).$$

The relevant SDE is

$$\begin{aligned} dY_N(t) &= \mathbf{i}\lambda S(T-t)\pi_N\Big[|S(t-T)Y_N(t)|^2 S(t-T)Y_N(t)\Big]dt + S(T-t)\pi_N Q^{\frac{1}{2}}dW \\ &:= H_N(Y_N(t))dt + S(T-t)\pi_N Q^{\frac{1}{2}}dW(t). \end{aligned}$$

Apply Itô's formula to $t \to v(T-t, Y_N(t))$ and we get

$$\begin{aligned} dv(T-t, Y_N(t)) &= \frac{\partial v}{\partial t}(T-t, Y_N(t))dt \\ &\quad + \Big(Dv(T-t, Y_N(t)), H_N(Y_N(t))dt + S(T-t)\pi_N Q^{\frac{1}{2}}dW(t)\Big) \\ &\quad + \frac{1}{2}Tr\Big[(S(T-t)\pi_N Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t))S(T-t)\pi_N Q^{\frac{1}{2}}\Big]dt \\ &= \Big(Dv(T-t, Y_N(t)), S(T-t)\pi_N Q^{\frac{1}{2}}dW(t)\Big) \\ &\quad + \Big(Dv(T-t, Y_N(t)), H_N\left(Y_N(t)\right) - H\left(Y_N(t)\right)\Big)dt \\ &\quad - \frac{1}{2}Tr\Big[(S(T-t)Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t))S(T-t)Q^{\frac{1}{2}}\Big]dt \\ &\quad + \frac{1}{2}Tr\Big[(S(T-t)\pi_N Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t))S(T-t)\pi_N Q^{\frac{1}{2}}\Big]dt. \end{aligned}$$

Therefore,

$$\begin{aligned}v(0, Y_N(T)) &= v(T, Y_N(0)) + \int_0^T \Big(Dv(T-s, Y_N(s)), S(T-s)\pi_N Q^{\frac{1}{2}} dW(s)\Big) \\ &+ \int_0^T \Big(Dv\big(T-t, Y_N(t)\big), H_N\big(Y_N(t)\big) - H\big(Y_N(t)\big)\Big)dt \\ &+ \frac{1}{2}\int_0^T Tr\left[(S(T-t)\pi_N Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t)) S(T-t)\pi_N Q^{\frac{1}{2}}\right] dt \\ &- \frac{1}{2}\int_0^T Tr\left[(S(T-t) Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t)) S(T-t) Q^{\frac{1}{2}}\right] dt. \quad (3.11)\end{aligned}$$

By the construction of Y_N, we can check that

$$Y_N(0) = S(T)\pi_N u_0 \quad \text{and} \quad Y_N(T) = u_N(T).$$

According to the representation of v, we have

$$v(0, Y_N(T)) = E\left[\phi(Y(T))|Y(T) = Y_N(T)\right] = E\left[\phi(u_N(T))|Y(T) = Y_N(T)\right]$$

and

$$\begin{aligned}v(T, Y_N(0)) &= E\left[\phi(Y(T))|Y(0) = S(T)\pi_N u_0\right] \\ &= E\Big[\phi\Big(S(T)\pi_N u_0 + \int_0^T H(Y(t))dt \\ &\quad + S(T-t)Q^{\frac{1}{2}} dW(t)\Big)\Big| Y(0) = S(T)\pi_N u_0\Big].\end{aligned}$$

Take expectation of the two sides of Eq. 3.11 and we get

$$\begin{aligned}E\left[\phi(u_N(T))\right] &= E\left[\phi\Big(S(T)\pi_N u_0 + \int_0^T H(Y(t))dt + S(T-t)Q^{\frac{1}{2}} dW(t)\Big)\right] \\ &+ E\int_0^T \Big(Dv\big(T-t, Y_N(t)\big), H_N\big(Y_N(t)\big) - H\big(Y_N(t)\big)\Big)dt \\ &+ \frac{1}{2}E\int_0^T \Big\{Tr\left[(S(T-t)\pi_N Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t)) S(T-t)\pi_N Q^{\frac{1}{2}}\right] \\ &- Tr\left[(S(T-t) Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t)) S(T-t) Q^{\frac{1}{2}}\right]\Big\} dt. \quad (3.12)\end{aligned}$$

Step 3. Weak error of the solutions.

Subtracting the resulting Eqs. 3.10 and 3.12 leads to

$$
\begin{aligned}
& E\left[\phi(u_N(T))\right] - E\left[\phi(u(T))\right] \\
= \; & E\Big[\phi\Big(S(T)\pi_N u_0 + \int_0^T H(Y(t))dt + S(T-t)Q^{\frac{1}{2}}dW(t)\Big) \\
& -\phi\Big(S(T)u_0 + \int_0^T H(Y(t))dt + S(T-t)Q^{\frac{1}{2}}dW(t)\Big)\Big] \\
& +E\int_0^T \Big(Dv\big(T-t, Y_N(t)\big), H_N\big(Y_N(t)\big) - H\big(Y_N(t)\big)\Big)dt \\
& +\frac{1}{2}E\int_0^T \Big\{Tr\Big[(S(T-t)\pi_N Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t))S(T-t)\pi_N Q^{\frac{1}{2}}\Big] \\
& -Tr\Big[(S(T-t)Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t))S(T-t)Q^{\frac{1}{2}}\Big]\Big\}dt \\
:= \; & I + II + III. \qquad (3.13)
\end{aligned}
$$

Due to Lemma 1, terms I and II can be estimated as

$$
|I| \le C\,\|\phi\|_{C_b^1}\, E\,\|S(T)u_0 - S(T)\pi_N u_0\|_0 \le Ce^{-\alpha T}\|\phi\|_{C_b^1} E\|u_0\|_2 N^{-2} \le Ce^{-\alpha T}N^{-2}, \qquad (3.14)
$$

and

$$
\begin{aligned}
|II| \; & \le \; CE\int_0^T \|\phi\|_{C_b^1}\|H_N(Y_N(t)) - H(Y_N(t))\|_0 dt \\
& = \; CE\int_0^T \|\phi\|_{C_b^1}\|\mathbf{i}\lambda S(T-t)(Id-\pi_N)\big(|u_N(t)|^2 u_N(t)\big)\|_0 dt \\
& \le \; |\lambda| C\int_0^T e^{-\alpha(T-t)}\|\phi\|_{C_b^1} E\Big[\|u_N(t)\|_1^2\|u_N(t)\|_2\Big]N^{-2}dt \\
& \le \; |\lambda|\frac{C}{\alpha}N^{-2} \qquad (3.15)
\end{aligned}
$$

based on Lemma 1, Proposition 3.1 and the embedding $H^1 \hookrightarrow L^\infty$ in $\mathbb{R}$. In the first step of Eq. 3.15, we have used the fact (3.7).

Let us now estimate term III. As $(S(T-t)\pi_N - S(T-t))Q^{\frac{1}{2}}$ is a bounded linear operator and so is $D^2 v$ shown in Eq. 3.8, we have

$$
\begin{aligned}
& \Big|Tr\Big[(S(T-t)\pi_N Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t))S(T-t)\pi_N Q^{\frac{1}{2}}\Big] \\
& -Tr\Big[(S(T-t)Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t))S(T-t)Q^{\frac{1}{2}}\Big]\Big| \\
= \; & \Big|Tr\Big[((S(T-t)\pi_N - S(T-t))Q^{\frac{1}{2}})^* D^2 v(T-t, Y_N(t))(S(T-t)\pi_N + S(T-t))Q^{\frac{1}{2}}\Big]\Big| \\
\le \; & C\|S(T-t)\pi_N - S(T-t)\|_{\mathcal{L}(\dot{H}^2, L^2)}\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2, \dot{H}^2)}\|\phi\|_{C_b^2}\|S(T-t)\|_{\mathcal{L}(L^2, L^2)}\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2, L^2)} \\
\le \; & Ce^{-\alpha(T-t)}N^{-2}.
\end{aligned}
$$

Hence, integrating above equation leads to

$$
|III| \le \frac{C}{\alpha}N^{-2}. \qquad (3.16)
$$

Plugging (3.14), (3.15) and (3.16) into (3.13), we get

$$\left|E\Big[\phi\big(u_N(T)\big)\Big] - E\Big[\phi\big(u(T)\big)\Big]\right| \leq C(e^{-\alpha T} + \frac{1}{\alpha})N^{-2} \leq CN^{-2}, \tag{3.17}$$

in which, C is independent of time T. □

3.4 Convergence Order between Invariant Measures μ and μ_N

Based on the ergodicity of stochastic processes u and u_N, for any deterministic $u_0 \in \dot{H}^2$, we have the following two equations

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E\phi\big(u(t)\big)dt = \int_{L^2} \phi(y)d\mu(y),$$
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E\phi\big(u_N(t)\big)dt = \int_{V_N} \phi(y)d\mu_N(y)$$

for any $\phi \in C_b^2(L^2)$. Due to the time-independence of the weak error in Theorem 3.2, it turns out for any fixed α and N,

$$\left|\int_{L^2} \phi(y)d\mu(y) - \int_{V_N} \phi(y)d\mu_N(y)\right| = \left|\lim_{T\to\infty} \frac{1}{T}\int_0^T E\phi\big(u(t)\big) - E\phi\big(u_N(t)\big)dt\right|$$
$$\leq \lim_{T\to\infty} \frac{1}{T}\int_0^T \left|E\phi\big(u(t)\big) - E\phi\big(u_N(t)\big)\right| dt \leq \lim_{T\to\infty} \frac{1}{T}\int_0^T C(e^{-\alpha t} + \frac{1}{\alpha})N^{-2}dt \leq \frac{C}{\alpha}N^{-2},$$

which implies that μ_N is a proper approximation of μ. Thus, we give the following theorem.

Theorem 3.3 *Assume that $u_0 \in \dot{H}^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^3)} < \infty$. The error between invariant measures μ and μ_N is of order 2, i.e.,*

$$\left|\int_{L^2} \phi(y)d\mu(y) - \int_{V_N} \phi(y)d\mu_N(y)\right| < \frac{C}{\alpha}N^{-2}.$$

Remark 3 Although the time-independent weak error between u and u_N is obtained under the assumption $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$, it is necessary to assume in addition $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^3)} < \infty$ in order to get the unique ergodicity of u (see [8]).

4 Full Discretization

In this section, we discretize (3.1) in temporal direction by a modification of the implicit Euler scheme to get a fully discrete scheme. We prove the ergodicity of the numerical solution u_N^k of the fully discrete scheme, and get weak order $\frac{1}{2}$ of u_N^k in temporal direction. Thus, we achieve at least the same order as the weak error for the error of invariant measure, as a result of the time-independency of the weak error and the ergodicity of the solution.

4.1 Fully Discrete Scheme

We use a modified implicit Euler scheme to approximate (3.1), and obtain the following scheme

$$\begin{cases} u_N^k - e^{-\alpha\tau} u_N^{k-1} = \left(\mathbf{i}\Delta u_N^k + \mathbf{i}\lambda\pi_N \left(\frac{|u_N^k|^2 + |e^{-\alpha\tau} u_N^{k-1}|^2}{2} u_N^k \right) \right) \tau + \pi_N Q^{\frac{1}{2}} \delta W_k \\ u_N^0 = \pi_N u_0(x), \end{cases} \tag{4.1}$$

where u_N^k is an approximation of $u_N(t_k)$, τ represents the uniform time step, $t_k = k\tau$, and $\delta W_k = W(t_k) - W(t_{k-1})$.

The well-posedness of scheme (4.1), together with the uniform boundedness of the numerical solution, is stated in the following proposition. The time step τ is assumed to satisfy $\alpha\tau \in [0, 1]$ in sequel.

Proposition 4.1 *Assume $u_0 \in \dot{H}^0$. For sufficiently small τ, there uniquely exists a family of V_N-valued and $\{\mathcal{F}_{t_k}\}_{k\in\mathbb{N}}$-adapted solutions $\{u_N^k\}_{k\in\mathbb{N}}$ of Eq. 4.1, which satisfies that for any integer $p \geq 2$, there exists a constant $C = C(p, \alpha, u_N^0) > 0$, such that*

$$E\|u_N^k\|_0^p \leq C, \quad \forall\, k \in \mathbb{N}.$$

Proof **Step 1.** Existence and uniqueness of solution.

Similar to [6], we fix a family $\{g_k\}_{k\in\mathbb{N}}$ of deterministic functions in V_N. We also fix $\tilde{u}_N^{k-1} \in V_N$, the existence of solution $\tilde{u}_N^k \in V_N$ of

$$\tilde{u}_N^k - e^{-\alpha\tau} \tilde{u}_N^{k-1} = \mathbf{i}\tau\Delta\tilde{u}_N^k + \mathbf{i}\lambda\tau\pi_N \left(\frac{|\tilde{u}_N^k|^2 + |e^{-\alpha\tau}\tilde{u}_N^{k-1}|^2}{2} \tilde{u}_N^k \right) + \sqrt{\tau} g_k \tag{4.2}$$

can be proved by using Brouwer fixed point theorem. Indeed, multiplying (4.2) by $\overline{\tilde{u}}_N^k$, integrating with respect to x and taking the real part, we get

$$\begin{aligned} &\|\tilde{u}_N^k\|_0^2 + \|\tilde{u}_N^k - e^{-\alpha\tau}\tilde{u}_N^{k-1}\|_0^2 - e^{-2\alpha\tau}\|\tilde{u}_N^{k-1}\|_0^2 \\ &= 2\sqrt{\tau} Re \left[\int_0^1 (\overline{\tilde{u}}_N^k - e^{-\alpha\tau}\overline{\tilde{u}}_N^{k-1}) g_k dx + \int_0^1 (e^{-\alpha\tau}\overline{\tilde{u}}_N^{k-1}) g_k dx \right] \\ &\leq \|\tilde{u}_N^k - e^{-\alpha\tau}\tilde{u}_N^{k-1}\|_0^2 + e^{-2\alpha\tau}\|\tilde{u}_N^{k-1}\|_0^2 + 2\tau\|g_k\|_0^2, \end{aligned}$$

i.e.,

$$\|\tilde{u}_N^k\|_0^2 \leq 2e^{-2\alpha\tau}\|\tilde{u}_N^{k-1}\|_0^2 + 2\tau\|g_k\|_0^2. \tag{4.3}$$

Define

$$\begin{aligned} \Lambda : V_N \times V_N &\to \mathcal{P}(L^2), \\ (\tilde{u}_N^{k-1}, g_k) &\mapsto \{\tilde{u}_N^k | \tilde{u}_N^k \text{ are solutions of (42)}\}, \end{aligned}$$

where $\mathcal{P}(L^2)$ is the power set of L^2. Equation 4.3 implies that Λ is continuous, and its graph is closed by the closed graph theorem. When the spaces are endowed with their Borel σ-algebras, there is a measurable continuous function $\kappa : V_N \times V_N \to L^2$ such that

$$\kappa(u, g) \in \Lambda(u, g), \ \forall\, (u, g) \in V_N \times V_N.$$

Assume that $u_N^{k-1} \in V_N$ is a $\mathcal{F}_{t_{k-1}}$-measurable random variable, then $u_N^k = \kappa(u_N^{k-1}, \frac{\pi_N Q^{\frac{1}{2}}\delta W_k}{\sqrt{\tau}})$ is an L^2-valued solution of Eq. 4.1. Moreover,

$$(1-\mathbf{i}\Delta\tau)u_N^k = e^{-\alpha\tau}u_N^{k-1} + \mathbf{i}\lambda\tau\pi_N\left(\frac{|u_N^k|^2 + |e^{-\alpha\tau}u_N^{k-1}|^2}{2}u_N^k\right) + \pi_N Q^{\frac{1}{2}}\delta W_k \in V_N.$$

Hence, u_N^k is actually a V_N-valued solution of Eq. 4.1.

For any given u_N^{k-1} and sufficiently small time step τ, the solution u_N^k is unique, which can be proved in a similar procedure as [2]. This fact will be used in proving the ergodicity of the numerical solution $\{u_N^k\}_{k\in\mathbb{N}}$, and it can be found in Appendix "The Proof of Uniqueness of the Solution for Eq. 4.1".

Step 2. Boundedness of the p-moments.

The constants C below may be different, but do not depend on time.

i) $p = 2$. To show the boundedness, we multiply (4.1) by $\overline{u}_N^k$, integrate in [0,1] with respect to the space variable, take expectation and take the real part,

$$E\|u_N^k\|_0^2 + E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2 - e^{-2\alpha\tau}E\|u_N^{k-1}\|_0^2 = 2ReE\int_0^1 \overline{u}_N^k\pi_N Q^{\frac{1}{2}}\delta W_k dx$$
$$= 2ReE\int_0^1 (\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})\pi_N Q^{\frac{1}{2}}\delta W_k dx \le E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2 + E\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_0^2.$$

It derives

$$\begin{aligned} E\|u_N^k\|_0^2 &\le e^{-2\alpha\tau}E\|u_N^{k-1}\|_0^2 + C\tau \le e^{-2\alpha\tau k}E\|u_N^0\|_0^2 + C\tau(1+e^{-2\alpha\tau}+\cdots+e^{-2\alpha\tau(k-1)}) \\ &\le e^{-2\alpha t_k}E\|u_N^0\|_0^2 + \frac{C\tau}{1-e^{-2\alpha\tau}} \le E\|u_N^0\|_0^2 + \frac{C}{e^{-1}2\alpha} \end{aligned}$$

for $\tau < \frac{1}{\alpha}$, where we have used $e^{-2\alpha\tau} < 1 - e^{-1}2\alpha\tau$ for $\tau < \frac{1}{\alpha}$.

ii) $p = 4$. In the case when p=2, without taking expectation, we have

$$\|u_N^k\|_0^2 - e^{-2\alpha\tau}\|u_N^{k-1}\|_0^2 + \|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2 = 2Re\int_0^1 \overline{u}_N^k\pi_N Q^{\frac{1}{2}}\delta W_k dx.$$

Multiply both sides by $\|u_N^k\|_0^2$, take expectation and take the real part and we get

$$\begin{aligned} (LHS) &= E\|u_N^k\|_0^4 - e^{-2\alpha\tau}E\|u_N^{k-1}\|_0^2\|u_N^k\|_0^2 + E\Big[\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2\|u_N^k\|_0^2\Big] \\ &= \frac{1}{2}\Big(E\|u_N^k\|_0^4 - e^{-4\alpha\tau}E\|u_N^{k-1}\|_0^4\Big) + \frac{1}{2}E\Big(\|u_N^k\|_0^2 - e^{-2\alpha\tau}\|u_N^{k-1}\|_0^2\Big)^2 \\ &\quad + E\Big[\|u_N^k - e^{-2\alpha\tau}u_N^{k-1}\|_0^2\|u_N^k\|_0^2\Big] \end{aligned}$$

and

$$
\begin{aligned}
(RHS) &= 2ReE\int_0^1 \|u_N^k\|_0^2 \overline{u}_N^k \pi_N Q^{\frac{1}{2}}\delta W_k dx \\
&= 2ReE\int_0^1 \left(\|u_N^k\|_0^2(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})\right)\pi_N Q^{\frac{1}{2}}\delta W_k dx \\
&\quad +2ReE\int_0^1 \left((\|u_N^k\|_0^2 - e^{-2\alpha\tau}\|u_N^{k-1}\|_0^2)e^{-\alpha\tau}\overline{u}_N^{k-1}\right)\pi_N Q^{\frac{1}{2}}\delta W_k dx \\
&\leq E\Big[\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2\|u_N^k\|_0^2\Big] + E\Big(\|u_N^k\|_0^2\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_0^2\Big) \\
&\quad +\frac{1}{4}E\Big(\|u_N^k\|_0^2 - e^{-2\alpha\tau}\|u_N^{k-1}\|_0^2\Big)^2 + 4e^{-2\alpha\tau}E\|\overline{u}_N^{k-1}\pi_N Q^{\frac{1}{2}}\delta W_k\|_0^2 \\
&\leq E\Big[\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2\|u_N^k\|_0^2\Big] + \frac{1}{2}E\Big(\|u_N^k\|_0^2 - e^{-2\alpha\tau}\|u_N^{k-1}\|_0^2\Big)^2 + C\tau.
\end{aligned}
$$

Compare (LHS) with (RHS), we obtain

$$
E\|u_N^k\|_0^4 \leq e^{-4\alpha\tau}E\|u_N^{k-1}\|_0^4 + C\tau \leq C.
$$

iii) $p = 3$. Using 1) and 2), it is easy to check that the following holds true

$$
E\|u_N^k\|_0^3 \leq E\frac{\|u_N^k\|_0^2 + \|u_N^k\|_0^4}{2} \leq C.
$$

iv) $p > 4$. By repeating above procedure, we complete the proof.

□

Before showing the weak error between $u_N(t)$ and u_N^k, we need some a priori estimates on $\|u_N^k\|_1$ and $\|u_N^k\|_2$.

Proposition 4.2 *Assume that $\lambda = 0$ or -1, $u_0 \in \dot{H}^1$, $u_N^0 = \pi_N u_0$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^1)} < \infty$. Then for any $p \geq 1$, there exists a constant $C = C(\alpha, u_0, p)$ independent of N and t_k, such that*

$$
E\mathcal{H}_k^p \leq C, \ \forall\, k \in \mathbb{N},
$$

where $\mathcal{H}_k := \|\nabla u_N^k\|_0^2 - \frac{\lambda}{2}\|u_N^k\|_{L^4}^4$.

Proof The proof for $\lambda = 0$ is in the same procedure as that for $\lambda = -1$ and is much easier. Here we only give the proof for $\lambda = -1$

$$
u_N^k - e^{-\alpha\tau}u_N^{k-1} = \left(\mathbf{i}\Delta u_N^k - \mathbf{i}\pi_N\left(\frac{|u_N^k|^2 + |e^{-\alpha\tau}u_N^{k-1}|^2}{2}u_N^k\right)\right)\tau + \pi_N Q^{\frac{1}{2}}\delta W_k. \quad (4.4)
$$

i) $p = 1$. Multiplying (4.4) by $\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1}$, integrating with respect to x, taking the imaginary part and using the fact $\big((Id - \pi_N)v, v_N\big) = 0, \ \forall\, v \in \dot{H}^0, \ v_N \in V_N$, we have

$$\begin{aligned}&\|\nabla u_N^k\|_0^2+\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2-e^{-2\alpha\tau}\|\nabla u_N^{k-1}\|_0^2\\=&\ -Re\int_0^1\Big(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\Big)u_N^k(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})dx\\&+\frac{2}{\tau}Im\int_0^1\pi_N Q^{\frac12}\delta W_k(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})dx\\=:&\ A+B.\end{aligned}$$

Simple computations yield

$$\begin{aligned}A&=-Re\left[\int_0^1\Big(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\Big)\Big(\frac{u_N^k+e^{-\alpha\tau}u_N^{k-1}}{2}+\frac{u_N^k-e^{-\alpha\tau}u_N^{k-1}}{2}\Big)(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})dx\right]\\&\le-\frac12\|u_N^k\|_{L^4}^4+\frac12 e^{-4\alpha\tau}\|u_N^{k-1}\|_{L^4}^4\le-\frac12\|u_N^k\|_{L^4}^4+\frac12 e^{-2\alpha\tau}\|u_N^{k-1}\|_{L^4}^4\end{aligned}$$

and

$$\begin{aligned}B&=\frac{2}{\tau}Im\left[\int_0^1\pi_N Q^{\frac12}\delta W_k\Big[-\mathbf{i}\tau\Delta\overline{u}_N^k+\mathbf{i}\tau\frac{|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2}{2}\overline{u}_N^k+\overline{\pi_N Q^{\frac12}\delta W_k}\Big]dx\right]\\&=2Re\left[\int_0^1\nabla(\pi_N Q^{\frac12}\delta W_k)\cdot\nabla\Big(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1}\Big)dx\right]+2Re\left[\int_0^1\nabla(\pi_N Q^{\frac12}\delta W_k)\cdot\nabla\Big(e^{-\alpha\tau}\overline{u}_N^{k-1}\Big)dx\right]\\&\quad+Re\left[\int_0^1\Big(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\Big)\overline{u}_N^k\cdot\pi_N Q^{\frac12}\delta W_k dx\right]\\&\le\frac14\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+C\|\nabla(\pi_N Q^{\frac12}\delta W_k)\|_0^2+2Re\left[\int_0^1\nabla(\pi_N Q^{\frac12}\delta W_k)\cdot\nabla\Big(e^{-\alpha\tau}\overline{u}_N^{k-1}\Big)dx\right]\\&\quad+Re\left[\int_0^1\Big(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\Big)\overline{u}_N^k\cdot\pi_N Q^{\frac12}\delta W_k dx\right].\end{aligned}$$

Denote $\mathcal{H}_k=\|\nabla u_N^k\|_0^2+\frac12\|u_N^k\|_{L^4}^4$, then

$$\begin{aligned}&E\mathcal{H}_k+\frac34 E\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2\\\le&\ e^{-2\alpha\tau}E\mathcal{H}_{k-1}+C\tau &(4.5)\\&+ReE\left[\int_0^1\Big(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\Big)\overline{u}_N^k\cdot\pi_N Q^{\frac12}\delta W_k dx\right]. &(4.6)\end{aligned}$$

Based on the formula

$$(|a|^2+|b|^2)\overline{a}=\overline{a}|a-b|^2+b(\overline{a}-\overline{b})^2+3|b|^2(\overline{a}-\overline{b})+\overline{b}|a-b|^2+(\overline{b})^2(a-b)+2|b|^2\overline{b},$$

the last term on the right hand side can be rewritten as

$$
\begin{aligned}
& ReE\left[\int_0^1 \left(|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2\right)\overline{u}_N^k\cdot\pi_N Q^{\frac{1}{2}}\delta W_k dx\right]\\
=\ & ReE\int_0^1 \overline{u}_N^k\left|u_N^k-e^{-\alpha\tau}u_N^{k-1}\right|^2\pi_N Q^{\frac{1}{2}}\delta W_k dx+ReE\int_0^1 e^{-\alpha\tau}u_N^{k-1}(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})^2\pi_N Q^{\frac{1}{2}}\delta W_k dx\\
& +3ReE\int_0^1 |e^{-\alpha\tau}u_N^{k-1}|^2(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})\pi_N Q^{\frac{1}{2}}\delta W_k dx\\
& +ReE\int_0^1 e^{-\alpha\tau}\overline{u}_N^{k-1}\left|u_N^k-e^{-\alpha\tau}u_N^{k-1}\right|^2\pi_N Q^{\frac{1}{2}}\delta W_k dx\\
& +ReE\int_0^1 (e^{-\alpha\tau}\overline{u}_N^{k-1})^2(u_N^k-e^{-\alpha\tau}u_N^{k-1})\pi_N Q^{\frac{1}{2}}\delta W_k dx+2ReE\int_0^1 |e^{-\alpha\tau}u_N^{k-1}|^2 e^{-\alpha\tau}\overline{u}_N^{k-1}\pi_N Q^{\frac{1}{2}}\delta W_k dx\\
=:\ & a+b+c+d+e+f.
\end{aligned}
$$

Noting that $f=0$, it suffices to estimate the other five terms

$$
\begin{aligned}
a+b+d \ \le\ & E\Big[\|u_N^k\|_0\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}\\
& +2\|e^{-\alpha\tau}u_N^{k-1}\|_0\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}\Big]\\
\le\ & E\left[\left(\|u_N^k\|_0+2\|e^{-\alpha\tau}u_N^{k-1}\|_0\right)\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^{\frac{1}{2}}\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^{\frac{3}{2}}\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}\right]\\
\le\ & \frac{1}{4}E\left[\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0\right]\\
& +CE\left[\left(\|u_N^k\|_0^2+\|e^{-\alpha\tau}u_N^{k-1}\|_0^2\right)\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^2\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}^2\right]\\
\le\ & \frac{1}{4}E\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+CE\left(\tau^{\frac{1}{2}}\left(\|u_N^k\|_0^2+\|e^{-\alpha\tau}u_N^{k-1}\|_0^2\right)\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^2\right)^2\\
& +CE\left(\tau^{-\frac{1}{2}}\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}^2\right)^2\\
\le\ & \frac{1}{4}E\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+C\tau,
\end{aligned}
$$

where in the last step we have used Proposition 4.1,

$$
\begin{aligned}
c+e \ \le\ & 4E\left[\|e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}\right]\\
\le\ & \frac{1}{2}E\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^2+8\eta\tau e^{-4\alpha\tau}E\|u_N^{k-1}\|_{L^4}^4\\
\le\ & \frac{1}{2}E\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^2+2E\left[\left(\sqrt{\alpha}\tau^{\frac{1}{2}}e^{-\alpha\tau}\|\nabla u_N^{k-1}\|_0\right)\left(\frac{C}{2\sqrt{\alpha}}8\eta\tau^{\frac{1}{2}}e^{-3\alpha\tau}\|u_N^{k-1}\|_0^3\right)\right]\\
\le\ & \frac{1}{2}E\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0^2+\alpha\tau e^{-2\alpha\tau}E\|\nabla u_N^{k-1}\|_0^2+C\tau.
\end{aligned}
$$

Then (4.5) turns to be

$$
E\mathcal{H}_k\le(1+\alpha\tau)e^{-2\alpha\tau}E\mathcal{H}_{k-1}+C\tau\le e^{-\alpha\tau}E\mathcal{H}_{k-1}+C\tau.
$$

We finally obtain that

$$
E\mathcal{H}_k\le C.
$$

ii) $p = 2$. From the case $p = 1$, by $\|\cdot\|_{L^4}^4 \le \|\nabla\cdot\|_0\|\cdot\|_0^3$, we get

$$\begin{aligned}
\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1} \le\ & C\|\nabla(\pi_N Q^{\frac12}\delta W_k)\|_0^2 \\
&+ C Re\left[\int_0^1 \nabla(\pi_N Q^{\frac12}\delta W_k)\cdot\nabla\Big(e^{-\alpha\tau}\overline{u}_N^{k-1}\Big)dx\right] \\
&+ C\left(\tau^{\frac12}\Big(\|u_N^k\|_0^2 + \|e^{-\alpha\tau}u_N^{k-1}\|_0^2\Big)\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^2\right)^2 \\
&+ C\left(\tau^{-\frac12}\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}^2\right)^2 + \alpha\tau e^{-2\alpha\tau}\mathcal{H}_{k-1} \\
&+ C\tau^{-1}\|u_N^{k-1}\|_0^6\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}^4.
\end{aligned}$$

Multiplying above formula by $\mathcal{H}_k$, we have

$$\begin{aligned}
& \mathcal{H}_k^2 + (\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1})^2 - e^{-4\alpha\tau}\mathcal{H}_{k-1}^2 \\
\le\ & C\mathcal{H}_k\|\nabla(\pi_N Q^{\frac12}\delta W_k)\|_0^2 + C\mathcal{H}_k Re\left[\int_0^1 \nabla(\pi_N Q^{\frac12}\delta W_k)\cdot\nabla\Big(e^{-\alpha\tau}\overline{u}_N^{k-1}\Big)dx\right] \\
& + C\tau\mathcal{H}_k\Big(\|u_N^k\|_0^2 + \|e^{-\alpha\tau}u_N^{k-1}\|_0^2\Big)^2\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^4 \\
& + C\mathcal{H}_k\left(\tau^{-\frac12}\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}^2\right)^2 + \alpha\tau e^{-2\alpha\tau}\mathcal{H}_k\mathcal{H}_{k-1} \\
& + C\tau^{-1}\mathcal{H}_k\|u_N^{k-1}\|_0^6\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}^4 \\
=:\ & a' + b' + c' + d' + e' + f',
\end{aligned}$$

where

$$\begin{aligned}
E[a'+b'+c'+d'] \le\ & \frac14 E(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1})^2 + C\tau \\
& + C\tau e^{-2\alpha\tau}E\left[\mathcal{H}_{k-1}\Big(\|u_N^k\|_0^2 + \|e^{-\alpha\tau}u_N^{k-1}\|_0^2\Big)^2\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^4\right] \\
\le\ & \frac14 E(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1})^2 + \frac12\tau e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2 + C\tau,
\end{aligned}$$

$$\begin{aligned}
E[e'] \le\ & \frac12 E\left(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1}\right)^2 + (\frac12\alpha^2\tau^2 + \alpha\tau)e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2 \\
\le\ & \frac12 E\left(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1}\right)^2 + \frac32\alpha\tau e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2
\end{aligned}$$

and

$$\begin{aligned}
E[f'] \le\ & \frac14 E\left(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1}\right)^2 + C\tau^{-2}E\left[\|u_N^{k-1}\|_0^{12}\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}^8\right] \\
& + \alpha\tau e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2 + C\tau^{-3}E\left[\|u_N^{k-1}\|_0^{12}\|\pi_N Q^{\frac12}\delta W_k\|_{L^\infty}^8\right] \\
\le\ & \frac14 E\left(\mathcal{H}_k - e^{-2\alpha\tau}\mathcal{H}_{k-1}\right)^2 + \alpha\tau e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2 + C\tau.
\end{aligned}$$

Then we conclude

$$E\mathcal{H}_k^2 \le (1+3\alpha\tau)e^{-4\alpha\tau}E\mathcal{H}_{k-1}^2 + C\tau \le e^{-\alpha\tau}E\mathcal{H}_{k-1}^2 + C\tau \le C,$$

where we have used $(1+3\alpha\tau)e^{-3\alpha\tau} \le 1$ for $\alpha\tau < 1$.

iii) For $p = 2^l$, $l \in \mathbb{N}$, the result can be proved by above procedure. So it also holds for any $p \in \mathbb{N}$.

□

Corollary 1 *Under the assumptions in Proposition 4.2, we have*

$$E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^{2p} \le C\tau^p,$$

where constant C is independent of N and t_k.

Proof It is easy to check this by multiplying $\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1}$ to both sides of Eq. 4.4, integrating with respect to x and taking expectation,

$$\begin{aligned}
& E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^{2p} \\
= \; & E\Bigg[\tau Im\int_0^1 \nabla u_N^k \nabla(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx + Re\int_0^1 \pi_N Q^{\frac{1}{2}}\delta W_k\left(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1}\right)dx \\
& + \frac{\tau}{4} Im\int_0^1 \left(|u_N^k|^2 + |e^{-\alpha\tau}u_N^{k-1}|^2\right)\left(u_N^k + e^{-\alpha\tau}u_N^{k-1}\right)\left(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1}\right)dx\Bigg]^p \\
\le \; & CE\Bigg[\tau^p\|\nabla u_N^k\|_0^p\|\nabla\left(u_N^k - e^{-\alpha\tau}u_N^{k-1}\right)\|_0^p \\
& + \tau^p\left(\|u_N^k\|_1^{2p} + \|u_N^{k-1}\|_1^{2p}\right)\left(\|u_N^k\|_0^{2p} + \|u_N^{k-1}\|_0^{2p}\right)\Bigg] \\
& + CE\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_0^{2p} + \frac{1}{2}E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^{2p} \\
\le \; & \frac{1}{2}E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^{2p} + C\tau^p.
\end{aligned}$$

Then we complete the proof by Proposition 4.2. □

Proposition 4.3 *Under the assumptions $\lambda = 0$ or -1, $u_0 \in \dot{H}^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$, we also have the uniform boundedness of 2-norm as follows*

$$E\|u_N^k\|_2^2 \le C, \ \forall k \in \mathbb{N},$$

where C is also independent of N and t_k.

Proof We also give the proof for $\lambda = -1$ only. Multiply (4.4) by $\Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})$, integrating with respect to x, and then taking the imaginary part, we obtain

$$\begin{aligned}
& \|\Delta u_N^k\|_0^2 + \|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0^2 - e^{-2\alpha\tau}\|\Delta u_N^{k-1}\|_0^2 \\
= \; & Re\int_0^1 \left(|u_N^k|^2 + |e^{-\alpha\tau}u_N^{k-1}|^2\right)u_N^k\Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
& - \frac{2}{\tau} Im\int_0^1 \pi_N Q^{\frac{1}{2}}\delta W_k \Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
=: \; & A' + B'.
\end{aligned}$$

According to the uniform boundedness of any order of 0-norm and 1-norm, we have the following estimations.

$$
\begin{aligned}
E[A'] &= ReE\int_0^1 |u_N^k|^2 u_N^k \Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&\quad + e^{-3\alpha\tau} ReE\int_0^1 |u_N^{k-1}|^2 u_N^{k-1} \Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&\quad + e^{-2\alpha\tau} ReE\int_0^1 |u_N^{k-1}|^2 (u_N^k - e^{-\alpha\tau}u_N^{k-1}) \Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&= ReE\int_0^1 |u_N^k|^2 u_N^k \Delta\overline{u}_N^k dx - e^{-4\alpha\tau} ReE\int_0^1 |u_N^{k-1}|^2 u_N^{k-1} \Delta\overline{u}_N^{k-1} dx \\
&\quad + e^{-2\alpha\tau} ReE\int_0^1 |u_N^{k-1}|^2 (u_N^k - e^{-\alpha\tau}u_N^{k-1}) \Delta(\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&\quad + ReE\int_0^1 u_N^k \Delta\overline{u}_N^k |u_N^k - e^{-\alpha\tau}u_N^{k-1}|^2 dx \\
&\quad + 2ReE\int_0^1 \overline{u}_N^k (\nabla u_N^k)^2 (\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&\quad + 4ReE\int_0^1 u_N^k |\nabla u_N^k|^2 (\overline{u}_N^k - e^{-\alpha\tau}\overline{u}_N^{k-1})dx \\
&\quad + ReE\int_0^1 (u_N^k - e^{-\alpha\tau}u_N^{k-1}) \Delta\overline{u}_N^k \left(|u_N^k|^2 - |e^{-\alpha\tau}u_N^{k-1}|^2\right) dx \\
&=: A_a^k - e^{-4\alpha\tau}A_a^{k-1} + A_b + A_c + A_d + A_e + A_f.
\end{aligned}
$$

We estimate above terms repectively and obtain

$$
\begin{aligned}
-e^{-4\alpha\tau}A_a^{k-1} &= -e^{-2\alpha\tau}A_a^{k-1} + e^{-2\alpha\tau}(1-e^{-2\alpha\tau})A_a^{k-1} \\
&\le -e^{-2\alpha\tau}A_a^{k-1} + C\tau E\|u_N^{k-1}\|_1^4 \le -e^{-2\alpha\tau}A_a^{k-1} + C\tau,
\end{aligned}
$$

$$
\begin{aligned}
A_b &\le e^{-2\alpha\tau} E\left[\|u_N^{k-1}\|_{L^\infty}^2 \|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0 \|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0\right] \\
&\le \frac{1}{6}E\|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0^2 + C\tau E\|u_N^{k-1}\|_1^8 + C\tau^{-1}E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^4 \\
&\le \frac{1}{6}E\|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0^2 + C\tau,
\end{aligned}
$$

$$
\begin{aligned}
A_c &\le E\left[\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2 \|u_N^k\|_{L^\infty} \|\Delta u_N^k\|_0\right] \\
&\le C\tau^{-1}E\left[\|\nabla(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0 \|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^3 \|u_N^k\|_1^2\right] + \frac{1}{8}\alpha\tau E\|\Delta u_N^k\|_0^2 \\
&\le \frac{1}{6}E\|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0^2 + C\tau^{-5}E\|u_N^k - e^{-\alpha\tau}u_N^{k-1}\|_0^{12} \\
&\quad + C\tau E\|u_N^k\|_1^8 + \frac{1}{8}\alpha\tau E\|\Delta u_N^k\|_0^2 \\
&\le \frac{1}{6}E\|\Delta(u_N^k - e^{-\alpha\tau}u_N^{k-1})\|_0^2 + \frac{1}{8}\alpha\tau E\|\Delta u_N^k\|_0^2 + C\tau,
\end{aligned}
$$

$$\begin{aligned}A_d &= 2ReE\int_0^1 \overline{u}_N^k(\nabla u_N^k)^2\Big[-\mathbf{i}\tau\Delta\overline{u}_N^k+\mathbf{i}\tau\pi_N\left(\frac{|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2}{2}\overline{u}_N^k\right)\\ &\quad+\overline{\pi_N Q^{\frac{1}{2}}\delta W_k}\Big]dx\\ &\le \frac{1}{16}\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau+2ReE\int_0^1\overline{u}_N^k(\nabla u_N^k)^2\overline{\pi_N Q^{\frac{1}{2}}\delta W_k}dx\\ &\le \frac{1}{16}\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau+2ReE\int_0^1(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})(\nabla u_N^k)^2\overline{\pi_N Q^{\frac{1}{2}}\delta W_k}dx\\ &\quad+2ReE\int_0^1 e^{-\alpha\tau}\overline{u}_N^{k-1}\left((\nabla u_N^k)^2-(e^{-\alpha\tau}\nabla u_N^{k-1})^2\right)\overline{\pi_N Q^{\frac{1}{2}}\delta W_k}dx\\ &\le \frac{1}{16}\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau+CE\left[\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_0\|\nabla u_N^k\|_{L^4}^2\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}\right]\\ &\quad+CE\left[\|\nabla(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0\left(\|u_N^{k-1}\|_1\|u_N^k\|_1+\|u_N^{k-1}\|_1^2\right)\|\pi_N Q^{\frac{1}{2}}\delta W_k\|_{L^\infty}\right]\\ &\le \frac{1}{6}E\|\Delta(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+\frac{1}{8}\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau,\end{aligned}$$

and

$$\begin{aligned}A_f &= ReE\int_0^1(u_N^k-e^{-\alpha\tau}u_N^{k-1})\Delta\overline{u}_N^k Re\Big[\left(u_N^k-e^{-\alpha\tau}u_N^{k-1}\right)(\overline{u}_N^k+e^{-\alpha\tau}\overline{u}_N^{k-1})\Big]dx\\ &\le E\left[\|u_N^k-e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2(\|u_N^k\|_{L^\infty}+\|u_N^{k-1}\|_{L^\infty})\|\Delta u_N^k\|_0\right]\\ &\le \frac{1}{6}E\|\Delta(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+\frac{1}{8}\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau,\end{aligned}$$

where A_e has an same estimation as A_d and we have used that $\|\nabla\cdot\|_0\cong\|\cdot\|_1\le\|\cdot\|_2\cong\|\Delta\cdot\|_0$. So we obtain

$$E[A']\le\frac{5}{6}E\|\Delta(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+\frac{1}{2}\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau.$$

For term B', we have

$$\begin{aligned}E[B'] &= -\frac{2}{\tau}ImE\int_0^1\Delta\left(\pi_N Q^{\frac{1}{2}}\delta W_k\right)\left(-\mathbf{i}\tau\Delta\overline{u}_N^k+\mathbf{i}\pi_N\left(\frac{|u_N^k|^2+|e^{-\alpha\tau}u_N^{k-1}|^2}{2}\overline{u}_N^k\right)\tau+\overline{\pi_N Q^{\frac{1}{2}}\delta W_k}\right)dx\\ &= 2ReE\int_0^1\Delta\left(\pi_N Q^{\frac{1}{2}}\delta W_k\right)\Delta(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})dx\\ &\quad-ReE\int_0^1\Delta\left(\pi_N Q^{\frac{1}{2}}\delta W_k\right)\left(|u_N^k|^2\overline{u}_N^k-|e^{-\alpha\tau}u_N^{k-1}|^2e^{-\alpha\tau}u_N^{k-1}\right)dx\\ &\quad-ReE\int_0^1\Delta\left(\pi_N Q^{\frac{1}{2}}\delta W_k\right)|e^{-\alpha\tau}u_N^{k-1}|^2(\overline{u}_N^k-e^{-\alpha\tau}\overline{u}_N^{k-1})dx\\ &\le \frac{1}{6}E\|\Delta(u_N^k-e^{-\alpha\tau}u_N^{k-1})\|_0^2+C\tau.\end{aligned}$$

Denoting $\mathcal{K}_k:=\|\Delta u_N^k\|_0^2-Re\int_0^1|u_N^k|^2u_N^k\Delta\overline{u}_N^kdx$, then $E\|\Delta u_N^k\|_0^2\le E\mathcal{K}_k+C$ and

$$E\mathcal{K}_k-e^{-2\alpha\tau}E\mathcal{K}_{k-1}\le\frac{1}{2}\alpha\tau E\|\Delta u_N^k\|_0^2+C\tau\le\frac{1}{2}\alpha\tau E\mathcal{K}_k+C\tau.$$

Finally,

$$E\mathcal{K}_k \le (1 - \frac{1}{2}\alpha\tau)^{-1} e^{-2\alpha\tau} E\mathcal{K}_{k-1} + C\tau \le C,$$

where we have used $(1 - \frac{1}{2}\alpha\tau)^{-1} e^{-2\alpha\tau} \le e^{-\alpha\tau}$ for $\alpha\tau < 1$. □

4.2 Ergodicity of the Fully Discrete Scheme

To prove the ergodicity of the scheme (4.1), we will use the discrete form of Theorem 2.1. We give some existing results before our theorem.

Assumption 1 (Minorization condition in [14]) *The Markov chain $(x_n)_{n\in\mathbb{N}}$ with transition kernel $P_n(x, G) = P(x_n \in G|x_0 = x)$ satisfies, for some fixed compact set $\mathcal{C} \in \mathcal{B}(\mathbb{R}^d)$, the following:*

i) *for some $y^* \in int(\mathcal{C})$ there is, for any $\delta > 0$, a $t_1 = t_1(\delta) \in \mathbb{N}$ such that*

$$P_{t_1}(x, B_\delta(y^*)) > 0 \quad \forall x \in \mathcal{C};$$

ii) *the transition kernel possesses a density $p_n(x, y)$, more precisely*

$$P_n(x, G) = \int_G p_n(x, y)dy \quad \forall x \in \mathcal{C},\ G \in \mathcal{B}(\mathbb{R}^d) \cap \mathcal{B}(\mathcal{C})$$

and $p_n(x, y)$ is jointly continuous in $(x, y) \in \mathcal{C} \times \mathcal{C}$.

Assumption 2 (Lyapunov condition in [14]) *There is a function $F : \mathbb{R}^d \to [1, \infty)$, with* $\lim_{|x|\to\infty} F(x) = \infty$*, real numbers $\theta \in (0, 1)$, and $\gamma \in [0, \infty)$ such that*

$$E[F(x_{n+1})|\mathcal{F}_n] \le \theta F(x_n) + \gamma.$$

Definition 3 We say that function F is essentially quadratic if there exist constants $C_i > 0,\ i = 1, 2, 3$, such that

$$C_1(1 + \|x\|^2) \le F(x) \le C_2(1 + \|x\|^2), \quad |\nabla F(x)| \le C_3(1 + \|x\|).$$

Theorem 4.1 ([14]) *Assume that a Markov chain $(x_n)_{n\in\mathbb{N}}$ satisfies Assumptions 1 and 2 with an essentially quadratic F, then the chain possesses a unique invariant measure.*

Based on the preliminaries above and the theory of Markov chains, we prove the following theorem.

Theorem 4.2 *For all τ sufficiently small, the solution $(u_N^k)_{k\in\mathbb{N}}$ of scheme (4.1) has a unique invariant measure μ_N^τ. Thus, it is ergodic.*

Proof i) Lyapunov condition. Based on Proposition 4.1, we can take essentially quadratic function $F(\cdot) = 1 + \|\cdot\|_0^2$ as the Lyapunov function, and the Lyapunov condition holds.

ii) Minorization condition. In scheme (4.1), it gives

$$P_N^k = e^{-\alpha\tau} P_N^{k-1} - \tau \left(\Delta Q_N^k + \frac{\lambda}{2}\pi_N \left(\left(|P_N^k|^2 + |Q_N^k|^2 + |e^{-\alpha\tau} P_N^{k-1}|^2 + |e^{-\alpha\tau} Q_N^{k-1}|^2 \right) Q_N^k \right) \right) + \sum_{m=1}^{N} \sqrt{\eta_m} e_m \delta_k \beta_m^1, \tag{4.7}$$

$$Q_N^k = e^{-\alpha\tau} Q_N^{k-1} + \tau \left(\Delta P_N^k + \frac{\lambda}{2}\pi_N \left(\left(|P_N^k|^2 + |Q_N^k|^2 + |e^{-\alpha\tau} P_N^{k-1}|^2 + |e^{-\alpha\tau} Q_N^{k-1}|^2 \right) P_N^k \right) \right) + \sum_{m=1}^{N} \sqrt{\eta_m} e_m \delta_k \beta_m^2, \tag{4.8}$$

where P_N^k and Q_N^k denote the real and imaginary part of u_N^k respectively, that is $u_N^k = P_N^k + \mathbf{i} Q_N^k$. Also, $\pi_N Q^{\frac{1}{2}} \delta W_k = \sum_{m=1}^N \sqrt{\eta_m} e_m \left(\delta_k \beta_m^1 + \mathbf{i} \delta_k \beta_m^2 \right)$, where $\delta_k \beta_m^1$ and $\delta_k \beta_m^2$ are the real and imaginary part of δW_k respectively.

For any $y_1 = a_1 + \mathbf{i} b_1$, $y_2 = a_2 + \mathbf{i} b_2 \in V_N$ with a_i and b_i denoting the real and imaginary part of y_i $(i = 1, 2)$ respectively, as $\{e_m\}_{m=1}^N$ is a basis of V_N, $\{\delta_k \beta_m^1, \ \delta_k \beta_m^2\}_{m=1}^N$ can be uniquely determined to ensure that $(P_N^{k-1}, Q_N^{k-1}) = (a_1, b_1)$ and $(P_N^k, Q_N^k) = (a_2, b_2)$, which implies the irreducibility of u_N^k.

As stated in Proposition 4.1, the $\mathcal{F}_{t_k}$-measurable solution $\{u_N^k\}_{k\in\mathbb{N}}$ is defined through a unique continuous function: $u_N^k = \kappa(u_N^{k-1}, \frac{\pi_N Q^{\frac{1}{2}} \delta W_k}{\sqrt{\tau}})$, where δW_k has a C^∞ density. Thus, the transition kernel $P_1(x, G)$, $G \in \mathcal{B}(V_N)$ possesses a jointly continuous density $p_1(x, y)$. Furthermore, densities $p_k(x, y)$ are achieved by the time-homogeneous property of Markov chain $\{u_N^k\}_{k\in\mathbb{N}}$.

With above conditions, based on Theorem 4.1, we prove that u_N^k possesses a unique invariant measure. □

4.3 Weak Error between Solutions u_N and u_N^k

We still use modified processes to calculate the weak error of the fully discrete scheme in temporal direction. Denote $S_\tau = (Id - \mathbf{i}\tau\Delta)^{-1} e^{-\alpha\tau}$, then scheme (4.1) is rewritten as

$$\begin{aligned} u_N^k &= S_\tau u_N^{k-1} + \mathbf{i}\lambda\tau e^{\alpha\tau} S_\tau \pi_N \left(\frac{|u_N^k|^2 + |e^{-\alpha\tau} u_N^{k-1}|^2}{2} u_N^k \right) + e^{\alpha\tau} S_\tau \pi_N Q^{\frac{1}{2}} \delta W_k \\ &= S_\tau^k u_N^0 + \mathbf{i}\lambda\tau e^{\alpha\tau} \sum_{l=1}^k S_\tau^{k+1-l} \pi_N \left(\frac{|u_N^l|^2 + |e^{-\alpha\tau} u_N^{l-1}|^2}{2} u_N^l \right) + e^{\alpha\tau} \sum_{l=1}^k S_\tau^{k+1-l} \pi_N Q^{\frac{1}{2}} \delta W_l \end{aligned} \tag{4.9}$$

Lemma 2 *For any $k \in \mathbb{N}$ and sufficiently small τ, we have the following estimates,*

$$i) \ \|S_\tau^k - S(t)\|_{\mathcal{L}(\dot{H}^2, L^2)} \le C(t+\tau)^{\frac{1}{2}} e^{-\alpha t} \tau^{\frac{1}{2}}, \quad t \in [t_{k-1}, t_{k+1}],$$
$$ii) \ \|S_\tau^k - S(t)\|_{\mathcal{L}(\dot{H}^1, \dot{H}^1)} \le C e^{-\alpha t}, \quad t \in [t_{k-1}, t_{k+1}],$$

where the constant $C = C(\alpha)$ is independent of k and τ.

Proof **Step 1.** If $t = t_k$. As $S(t)$ is the operator semigroup of equation $du(t) = (\mathbf{i}\Delta - \alpha)u(t)dt$, $u(0) = u^0 \in \dot{H}^2$, and S_τ is the corresponding discrete operator semigroup, we have

$$S_\tau^k u(0) = u^k = e^{-\alpha\tau} u^{k-1} + \mathbf{i}\tau\Delta u^k, \tag{4.10}$$

$$S(t_k)u(0) = u(t_k) = e^{-\alpha\tau} u(t_{k-1}) + \int_{t_{k-1}}^{t_k} \mathbf{i} e^{-\alpha(t_k - s)} \Delta u(s) ds. \tag{4.11}$$

Denote $e_k = u^k - u(t_k) = \left(S_\tau^k - S(t_k)\right) u(0)$ with $e_0 = 0$, then

$$e_k = e^{-\alpha\tau} e_{k-1} + \mathbf{i}\tau\Delta e_k + \mathbf{i}\int_{t_{k-1}}^{t_k} \left[\Delta u(t_k) - e^{-\alpha(t_k - s)}\Delta u(s)\right] ds.$$

Multiply $\overline{e}_k$ to above formula, integrate with respect to x, take the real part, and we get

$$\begin{aligned}
&\frac{1}{2}\left[\|e_k\|_0^2 + \|e_k - e^{-\alpha\tau} e_{k-1}\|_0^2 - e^{-2\alpha\tau}\|e_{k-1}\|_0^2\right] \\
&= Re\left[\mathbf{i}\int_0^1 \int_{t_{k-1}}^{t_k} \Delta\overline{e}_k \int_s^{t_k} \mathbf{i} e^{-\alpha(t_k - r)} \Delta u(r) dr ds dx\right] \\
&\leq C \int_{t_{k-1}}^{t_k} \int_s^{t_k} \|\Delta u^k - \Delta u(t_k)\|_0 \|\Delta u(r)\|_0 dr ds \\
&\leq C e^{-2\alpha t_k} \|\Delta u(0)\|_0^2 \tau^2,
\end{aligned}$$

where we have used the fact that $\|\Delta u^k\|_0^2 \leq e^{-2\alpha t_k}\|\Delta u^0\|_0^2$ and $\|\Delta u(t)\|_0 \leq Ce^{-\alpha t}\|\Delta u(0)\|_0$. In fact, multiplying $\Delta\overline{u}^k - e^{-\alpha\tau}\Delta\overline{u}^{k-1}$ to Eq. 4.10, integrating in space and taking the imaginary part, we obtain

$$\|\Delta u^k\|_0^2 \leq e^{-2\alpha\tau}\|\Delta u^{k-1}\|_0^2 \leq e^{-2\alpha t_k}\|\Delta u^0\|_0^2.$$

Then it's easy to check that

$$\|e_k\|_0^2 \leq e^{-2\alpha\tau}\|e_{k-1}\|_0^2 + Ce^{-2\alpha t_k}\|\Delta u(0)\|_0^2 \tau^2$$

leads to

$$\|e_k\|_0^2 \leq C t_k e^{-2\alpha t_k}\|\Delta u(0)\|_0^2 \tau, \tag{4.12}$$

which finally yields $\|S_\tau^k - S(t_k)\|_{\mathcal{L}(\dot{H}^2, L^2)} \leq C t_k^{\frac{1}{2}} e^{-\alpha t_k} \tau^{\frac{1}{2}}$ in i).

For ii), we have

$$\begin{aligned}
\|\left(S_\tau^k - S(t_k)\right) u(0)\|_1^2 &= \sum_{n=1}^{\infty}\left|e^{-\alpha t_k}\left((1 + n^2\pi^2)^{-k} - e^{-n^2\pi^2 t_k}\right)(u(0), e_n)\right|^2 |\lambda_n| \\
&\leq 4e^{-2\alpha t_k}\sum_{n=1}^{\infty}|(u(0), e_n)|^2 |\lambda_n| = 4e^{-2\alpha t_k}\|u(0)\|_1^2.
\end{aligned}$$

In the following two steps, we only give the proof of i), and ii) can be proved in a same procedure. We use the notation $\|\cdot\| = \|\cdot\|_{\mathcal{L}(\dot{H}^2,L^2)}$, which is an operator norm defined at the beginning of this paper.

Step 2. If $t \in [t_{k-1}, t_k]$,

$$\begin{aligned}\|S_\tau^k - S(t)\| &\leq \|S_\tau^k - S(t_k)\| + \|S(t_k) - S(t)\| \leq Ct_k^{\frac{1}{2}} e^{-\alpha t_k} \tau^{\frac{1}{2}} \\ &\quad + e^{-\alpha t} |e^{-\alpha(t_k - t)} - 1| \\ &\leq Ct_k^{\frac{1}{2}} e^{-\alpha t_k} \tau^{\frac{1}{2}} + e^{-\alpha t} \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha\tau)^n \leq Ct_k^{\frac{1}{2}} e^{-\alpha t_k} \tau^{\frac{1}{2}} \\ &\quad + e^{-\alpha t} \alpha\tau \frac{e^{\alpha\tau} - 1}{\alpha\tau} \\ &\leq C(t+\tau)^{\frac{1}{2}} e^{-\alpha t} \tau^{\frac{1}{2}}.\end{aligned}$$

We have used the fact that $\frac{e^{\alpha\tau}-1}{\alpha\tau}$ is uniformly bounded for $\alpha\tau \in [0, 1]$.

Step 3. If $t \in [t_k, t_{k+1}]$,

$$\begin{aligned}\|S_\tau^k - S(t)\| &\leq \|S_\tau^k - S(t_k)\| + \|S(t_k) - S(t)\| \leq Ct_k^{\frac{1}{2}} e^{-\alpha t_k} \tau^{\frac{1}{2}} \\ &\quad + e^{-\alpha t} |e^{-\alpha(t_k - t)} - 1| \\ &\leq Ct_k^{\frac{1}{2}} e^{-\alpha t} e^{\alpha(t-t_k)} \tau^{\frac{1}{2}} + e^{-\alpha t} \alpha\tau \frac{e^{\alpha\tau - 1}}{\alpha\tau} \leq C(t+\tau)^{\frac{1}{2}} e^{-\alpha t} \tau^{\frac{1}{2}}.\end{aligned}$$

We have used the fact $e^{\alpha(t-t_k)} \leq e^{\alpha\tau} \leq e$.

□

Remark 4 From Eq. 4.10, we can also prove that

$$\|S_\tau^k\|_{\mathcal{L}(L^2,L^2)} \leq Ce^{-\alpha t},$$

where k and t satisfying $t \in [t_{k-1}, t_{k+1}]$.

Next theorem gives the time-independent weak error of the solutions for different cases.

Theorem 4.3 *Assume that $u_0 \in \dot{H}^2$, $u_N^0 = u_N(0) = \pi_N u_0$ and $\|Q^{\frac{1}{2}}\|^2_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$. For the cases $\lambda = 0$ or -1, the weak errors are independent of time and of order $\frac{1}{2}$. That is, for any $\phi \in C_b^2(L^2)$, there exists a constant $C = C(u_0, \phi)$ independent of N, T and M, such that for any $T = M\tau$,*

$$\left| E[\phi(u_N(T))] - E[\phi(u_N^M)] \right| \leq C\tau^{\frac{1}{2}}.$$

Corollary 2 *Under above assumptions, for any $t \in [(M-1)\tau, (M+1)\tau]$, it also holds*

$$\left| E[\phi(u_N(t))] - E[\phi(u_N^M)] \right| \leq C\tau^{\frac{1}{2}}.$$

Proof Let $T = M\tau$. As

$$\left| E[\phi(u_N(t))] - E[\phi(u_N^M)] \right| = \left| E[\phi(u_N(T))] - E[\phi(u_N(t))] \right| + \left| E[\phi(u_N(T))] - E[\phi(u_N^M)] \right|$$

and

$$\left|E[\phi(u_N(T))] - E[\phi(u_N(t))]\right| \le \|\phi\|_{C_b^1} E\|u_N(T) - u_N(t)\|_0$$
$$\le \|\phi\|_{C_b^1}(T-t)\sup_{t\ge 0}\left[E\|u_N(t)\|_2 + E\|u_N(t)\|_0 + E\|u_N(t)\|_1^2\|u_N(t)\|_0\right]$$
$$+\|\phi\|_{C_b^1} E\|\pi_N Q^{\frac{1}{2}}\big(W(T) - W(t)\big)\|_0 \le C\tau^{\frac{1}{2}},$$

we then complete the proof according to Theorem 4.3. □

Proof of Theorem 4.3 We split it into several steps.

Step 1. Calculation of $E[\phi(u_N(T))]$.

Recall the process we constructed in the proof of Theorem 3.2,

$$dY_N(t) = H_N(Y_N(t))dt + S(T-t)\pi_N Q^{\frac{1}{2}} dW(t).$$

Now we denote $v_N(T-t, y) = E[\phi(Y_N(T))|Y_N(t) = y]$, then

$$v_N(0, Y_N(T)) = v_N(T, Y_N(0)) + \int_0^T \Big(Dv_N(T-t, Y_N(t)), S(T-t)\pi_N Q^{\frac{1}{2}} dW(t)\Big), \tag{4.13}$$

where

$$\begin{aligned} v_N(0, Y_N(T)) &= E[\phi(u_N(T))|Y_N(T) = u_N(T)],\\ v_N(T, Y_N(0)) &= E[\phi(Y_N(T))|Y_N(0) = S(T)u_N(0)]\\ &= E\left[\phi\left(S(T)u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\right| Y_N \\ &= S(T)u_N(0)\Big]. \end{aligned} \tag{4.14}$$

The expectation of Eq. 4.13 implies,

$$E[\phi(u_N(T))] = E\left[\phi\left(S(T)u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\right]. \tag{4.15}$$

Step 2. Calculation of $E[\phi(u_N^M)]$.

Similar to [9], we define a discrete modified process

$$\begin{aligned} Y_N^k &:= S_\tau^{M-k} u_N^k \\ &= S_\tau^M u_N^0 + \mathbf{i}\lambda\tau e^{\alpha\tau}\sum_{l=1}^k S_\tau^{M+1-l}\pi_N\left(\frac{|u_N^l|^2 + |e^{-\alpha\tau}u_N^{l-1}|^2}{2}u_N^l\right) \quad (4.16)\\ &\quad + e^{\alpha\tau}\sum_{l=1}^k S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\delta W_l \\ &= S_\tau^M u_N^0 + \mathbf{i}\lambda\tau e^{\alpha\tau}\sum_{l=1}^k S_\tau^{M+1-l}\pi_N\left(\frac{|S_\tau^{l-M}Y_N^l|^2 + |e^{-\alpha\tau}S_\tau^{l-1-M}Y_N^{l-1}|^2}{2}S_\tau^{l-M}Y_N^l\right) \quad (4.17)\\ &\quad + e^{\alpha\tau}\sum_{l=1}^k S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\delta W_l. \end{aligned}$$

Consider the following time continuous interpolation of Y_N^k, which is also V_N-valued and $\{\mathcal{F}_t\}_{t\geq 0}$-adaped,

$$\begin{aligned}\tilde{Y}_N(t) &:= S_\tau^M u_N^0 + \mathbf{i}\lambda e^{\alpha\tau}\int_0^t \sum_{l=1}^M S_\tau^{M+1-l}\pi_N\left(\frac{|S_\tau^{l-M}Y_N^l|^2 + |e^{-\alpha\tau}S_\tau^{l-1-M}Y_N^{l-1}|^2}{2}S_\tau^{l-M}Y_N^l\right)1_l(s)ds \\ &\quad + e^{\alpha\tau}\int_0^t \sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(s)dW(s) \\ &=: S_\tau^M u_N^0 + \int_0^t H_\tau(Y_N^M, s)ds + e^{\alpha\tau}\int_0^t \sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(s)dW(s).\end{aligned}$$

In particular for $t \in [t_{l-1}, t_l]$,

$$\begin{aligned}\tilde{Y}_N(t) &= Y_N^{l-1} + \mathbf{i}\lambda e^{\alpha\tau}S_\tau^{M+1-l}\pi_N\left(\frac{|S_\tau^{l-M}Y_N^l|^2 + |e^{-\alpha\tau}S_\tau^{l-1-M}Y_N^{l-1}|^2}{2}S_\tau^{l-M}Y_N^l\right)(t - t_{l-1}) \\ &\quad + e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\Big(W(t) - W(t_{l-1})\Big),\end{aligned} \tag{4.18}$$

or equivalently,

$$\begin{aligned}\tilde{Y}_N(t) &= Y_N^{l} + \mathbf{i}\lambda e^{\alpha\tau}S_\tau^{M+1-l}\pi_N\left(\frac{|S_\tau^{l-M}Y_N^l|^2 + |e^{-\alpha\tau}S_\tau^{l-1-M}Y_N^{l-1}|^2}{2}S_\tau^{l-M}Y_N^l\right)(t - t_{l}) \\ &\quad + e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\Big(W(t) - W(t_{l})\Big).\end{aligned} \tag{4.19}$$

Apply Itô's formula to $t \mapsto v_N(T-t, \tilde{Y}_N(t))$,

$$\begin{aligned}& dv_N(T-t, \tilde{Y}_N(t)) \\ &= \frac{\partial v_N}{\partial t}(T-t, \tilde{Y}_N(t))dt + \Big(Dv_N, H_\tau(Y_N^M, t)dt + e^{\alpha\tau}\sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(t)dW(t)\Big) \\ &\quad + \frac{1}{2}Tr\left[\left(e^{\alpha\tau}\sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(t)\right)^* D^2 v_N\left(e^{\alpha\tau}\sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(t)\right)\right]dt \\ &= \Big(Dv_N, H_\tau(Y_N^M, t) - H_N(\tilde{Y}_N(t))\Big)dt + \left(Dv_N, e^{\alpha\tau}\sum_{l=1}^M S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}1_l(t)dW(t)\right) \\ &\quad + \frac{1}{2}\sum_{l=1}^M Tr\left[\left(e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N\left(e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)\right]1_l(t)dt \\ &\quad - \frac{1}{2}\sum_{l=1}^M Tr\left[\left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N\left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\right]1_l(t)dt,\end{aligned}$$

where Dv_N and D^2v_N are evaluated at $(T-t, \tilde{Y}_N(t))$.

The same as before, integrate the formula above from 0 to T, and take expectation based on the fact that

$$
\begin{aligned}
v_N(0, \tilde{Y}_N(T)) &= E[\phi(Y_N(T))|Y_N(T) = \tilde{Y}_N(T)] = E[\phi(u_N^M)|Y_N(T) = u_N^M],\\
v_N(T, \tilde{Y}_N(0)) &= E[\phi(Y_N(T))|Y_N(0) = \tilde{Y}_N(0)]\\
&= E\left[\phi\left(S_\tau^M u_N(0) + \int_0^T H_N(Y_N(s))ds\right.\right.\\
&\qquad \left.\left. + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right) \middle| Y_N(0) = S_\tau^M u_N(0)\right],
\end{aligned}
$$

we get

$$
\begin{aligned}
E[\phi(u_N^M)] &= E\left[\phi\left(S_\tau^M u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\right]\\
&+ E\int_0^T \left(Dv_N, H_\tau(Y_N^M, t) - H_N(\tilde{Y}_N(t))\right) dt\\
&+ \frac{1}{2}\sum_{l=1}^M E\int_0^T Tr\Big[\left(e^{\alpha\tau} S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(e^{\alpha\tau} S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)\\
&- \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\Big] 1_l(t)dt.
\end{aligned}
\tag{4.20}
$$

Step 3. Weak convergence order.

Subtracting (4.15) from (4.20), we derive

$$
\begin{aligned}
&E[\phi(u_N^M)] - E[\phi(u_N(T))]\\
= \;& E\Big[\phi\left(S_\tau^M u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\\
&-\phi\left(S(T)u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\Big]\\
&+E\int_0^T \left(Dv_N, H_\tau(Y_N^M, t) - H_N(\tilde{Y}_N(t))\right) dt\\
&+\frac{1}{2}\sum_{l=1}^M E\int_0^T Tr\Big[\left(e^{\alpha\tau} S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(e^{\alpha\tau} S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)\\
&-\left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\Big] 1_l(t)dt.\\
=:\;& I + II + III.
\end{aligned}
$$

Now we estimate I, II, and III separately. The constants C below may be different but are all independent of T and τ.

$$\begin{aligned}
|I| &= \Bigg| E\left[\phi\left(S_\tau^M u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\right] \\
&\quad - E\left[\phi\left(S(T)u_N(0) + \int_0^T H_N(Y_N(s))ds + \int_0^T S(T-s)\pi_N Q^{\frac{1}{2}} dW\right)\right]\Bigg| \\
&\leq C\|\phi\|_{C_b^1} \|S_\tau^M u_N(0) - S(T)u_N(0)\|_0 \\
&\leq C\|\phi\|_{C_b^1} \|S_\tau^M - S(T)\|_{\mathcal{L}(\dot{H}^2, L^2)} \|u_N(0)\|_2 \\
&\leq C(T+\tau)^{\frac{1}{2}} e^{-\alpha T} \tau^{\frac{1}{2}},
\end{aligned} \tag{4.21}$$

where we have used Lemma 2 and $u_N(0) = \pi_N u_0 \in \dot{H}^2$.

Noticing $II = 0$ for $\lambda = 0$, now we consider the nonlinear term II for $\lambda = -1$. By using the notation $a_l := S_\tau^{l-M} Y_N^l = u_N^l$ and Eqs. 4.18 and 4.19, we can define b_l in two ways,

$$\begin{aligned}
b_l &:= S(t-T)\tilde{Y}_N(t) 1_l(t) \\
&= S(t-T) S_\tau^{M+1-l} u_N^{l-1} + e^{\alpha\tau} S(t-T) S_\tau^{M+1-l} \Bigg(\mathbf{i}\lambda \pi_N \left(\frac{|e^{-\alpha\tau} u_N^{l-1}|^2 + |u_N^l|^2}{2} u_N^l \right)(t - t_{l-1}) \\
&\quad + \pi_N Q^{\frac{1}{2}} (W(t) - W(t_{l-1}))\Bigg),
\end{aligned}$$

or equivalently,

$$\begin{aligned}
b_l &:= S(t-T)\tilde{Y}_N(t) 1_l(t) \\
&= S(t-T) S_\tau^{M-l} u_N^{l} + e^{\alpha\tau} S(t-T) S_\tau^{M+1-l} \Bigg(\mathbf{i}\lambda \pi_N \left(\frac{|e^{-\alpha\tau} u_N^{l-1}|^2 + |u_N^l|^2}{2} u_N^l \right)(t - t_{l}) \\
&\quad + \pi_N Q^{\frac{1}{2}} (W(t) - W(t_{l}))\Bigg).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& a_{l-1} - b_l \\
&= \left(Id - S(t-T) S_\tau^{M+1-l}\right) u_N^{l-1} \\
&\quad - e^{\alpha\tau} S(t-T) S_\tau^{M+1-l} \Bigg(\mathbf{i}\lambda \pi_N \left(\frac{|e^{-\alpha\tau} u_N^{l-1}|^2 + |u_N^l|^2}{2} u_N^l \right)(t - t_{l-1}) \\
&\quad + \pi_N Q^{\frac{1}{2}} (W(t) - W(t_{l-1}))\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
a_l - b_l &= \left(Id - S(t-T) S_\tau^{M-l}\right) u_N^{l} \\
&\quad - e^{\alpha\tau} S(t-T) S_\tau^{M+1-l} \Bigg(\mathbf{i}\lambda \pi_N \left(\frac{|e^{-\alpha\tau} u_N^{l-1}|^2 + |u_N^l|^2}{2} u_N^l \right) \\
&\quad + \pi_N Q^{\frac{1}{2}} (W(t) - W(t_{l}))\Bigg),
\end{aligned}$$

where $\|S(t-T)S_\tau^{M+1-l}\|_{\mathcal{L}(L^2,L^2)} \leq C$ and

$$\|Id - S(t-T)S_\tau^{M-l}\|_{\mathcal{L}(\dot{H}^2,L^2)} \leq \|S(t-T)\|_{\mathcal{L}(L^2,L^2)}\|S(T-t) - S_\tau^{M-l}\|_{\mathcal{L}(\dot{H}^2,L^2)} \leq C(T-t+\tau)^{\frac{1}{2}}\tau^{\frac{1}{2}}$$

according to Lemma 2s. Thus, we have the following estimate

$$\|a_l - b_l\|_0 \leq C\Big[(T-t+\tau)^{\frac{1}{2}}\tau^{\frac{1}{2}}\|u_N^l\|_2 + \tau\Big(\|u_N^{l-1}\|_1^2 + \|u_N^l\|_1^2\Big)\|u_N^l\|_0 + \|\pi_N Q^{\frac{1}{2}}(W(t) - W(t_l))\|_0\Big].$$

Also, $\|a_{l-1} - b_l\|_0$ can be estimated in the same way. Thus, based on Eq. 3.7, we have

$$|II| = \left|E\int_0^T \Big(Dv_N, H_\tau(Y_N^M, t) - H_N(\tilde{Y}_N(t))\Big)\,dt\right| \leq C\|\phi\|_{C_b^1}\int_0^T E\|H_\tau(Y_N^M, t) - H_N(\tilde{Y}_N(t))\|_0 dt, \tag{4.22}$$

where

$$\begin{aligned}
&H_\tau(Y_N^M, t) - H_N(\tilde{Y}_N(t)) \\
&= \sum_{l=1}^M \left[e^{\alpha\tau}S_\tau^{M+1-l}\pi_N\left(\mathbf{i}\lambda\frac{|e^{-\alpha\tau}a_{l-1}|^2 + |a_l|^2}{2}a_l\right) - S(T-t)\pi_N\Big(\mathbf{i}\lambda|b_l|^2 b_l\Big)\right]1_l(t) \\
&= \frac{\lambda}{2}\mathbf{i}\sum_{l=1}^M\Big[e^{\alpha\tau}\Big(S_\tau^{M+1-l} - S(T-t)\Big)\pi_N\Big(|e^{-\alpha\tau}a_{l-1}|^2 a_l\Big) \\
&\quad + (e^{-\alpha\tau} - 1)S(T-t)\pi_N\Big(|a_{l-1}|^2 a_l\Big) \\
&\quad + S(T-t)\pi_N\Big(|a_{l-1}|^2 a_l - |b_l|^2 b_l\Big)\Big]1_l(t) \\
&\quad + \frac{\lambda}{2}\mathbf{i}\sum_{l=1}^M\Big[e^{\alpha\tau}\Big(S_\tau^{M+1-l} - S(T-t)\Big)\pi_N\Big(|a_l|^2 a_l\Big) + (e^{\alpha\tau} - 1)S(T-t)\pi_N\Big(|a_l|^2 a_l\Big) \\
&\quad + S(T-t)\pi_N\Big(|a_l|^2 a_l - |b_l|^2 b_l\Big)\Big]1_l(t) \\
&= \frac{\lambda}{2}\mathbf{i}\Big[\sum_{l=1}^M e^{\alpha\tau}\Big(S_\tau^{M+1-l} - S(T-t)\Big)\pi_N\Big(|e^{-\alpha\tau}a_{l-1}|^2 a_l\Big)1_l(t) \\
&\quad + \sum_{l=1}^M S(T-t)\pi_N\Big(|a_{l-1}|^2(a_l - b_l)\Big)1_l(t) \\
&\quad + \sum_{l=1}^M S(T-t)\pi_N\Big(|b_l|^2(a_{l-1} - b_l)\Big)1_l(t) + \sum_{l=1}^M S(T-t)\pi_N\Big(a_{l-1}b_l(\overline{a}_{l-1} - \overline{b}_l)\Big)1_l(t) \\
&\quad + \sum_{l=1}^M (e^{-\alpha\tau} - 1)S(T-t)\pi_N\Big(|a_{l-1}|^2 a_l\Big)1_l(t) \\
&\quad + \sum_{l=1}^M e^{\alpha\tau}\Big(S_\tau^{M+1-l} - S(T-t)\Big)\pi_N\Big(|a_l|^2 a_l\Big)1_l(t)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{M} S(T-t)\pi_N\Big(|a_l|^2\,(a_l-b_l)\Big)1_l(t)+\sum_{l=1}^{M} S(T-t)\pi_N\Big(|b_l|^2(a_l-b_l)\Big)1_l(t)\\
&+\sum_{l=1}^{M} S(T-t)\pi_N\Big(a_l b_l(\overline{a}_l-\overline{b}_l)\Big)1_l(t)\Bigg]+\sum_{l=1}^{M}(e^{\alpha\tau}-1)S(T-t)\pi_N\Big(|a_l|^2 a_l\Big)1_l(t)\\
:=\;&\frac{\lambda}{2}\mathbf{i}\Big[II_1^{l-1}+II_2^{l-1}+II_3^{l-1}+II_4^{l-1}+II_5^{l-1}+II_1^{l}+II_2^{l}+II_3^{l}+II_4^{l}+II_5^{l}\Big].
\end{aligned}
$$

If $\lambda = -1$, thanks to the uniform estimations of 0-norm, 1-norm and 2-norm of u_N^k, we have the following estimates.

By the embedding $H^1 \hookrightarrow L^\infty$ in $\mathbb{R}^1$, we have following exponential estimates

$$
\begin{aligned}
E\|II_1^{l-1}\|_0 &\le \frac{1}{2}\sum_{l=1}^{M}\|S_\tau^{M+1-l}-S(T-t)\|_{\mathcal{L}(\dot{H}^2,L^2)}E\left\|\pi_N\left(|e^{-\alpha\tau}u_N^{l-1}|^2u_N^l\right)\right\|_2 1_l(t)\\
&\le C\sum_{l=1}^{M}\|S_\tau^{M+1-l}-S(T-t)\|_{\mathcal{L}(\dot{H}^2,L^2)}E\left[\|u_N^{l-1}\|_1^4+\|u_N^l\|_2^2\right]1_l(t)\\
&\le C(T-t+\tau)^{\frac{1}{2}}e^{-\alpha(T-t)}\tau^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
E\|II_2^{l-1}\|_0 &\le Ce^{-\alpha(T-t)}E\sum_{l=1}^{M}\|a_{l-1}\|_1^2\|a_l-b_l\|_0 1_l(t)\\
&\le \quad Ce^{-\alpha(T-t)}E\sum_{l=1}^{M}\|u_N^{l-1}\|_1^2\Bigg[C(T-t+\tau)^{\frac{1}{2}}\tau^{\frac{1}{2}}\|u_N^l\|_2\\
&\quad +C\left[\left(\|u_N^{l-1}\|_1^2+\|u_N^l\|_1^2\right)\|u_N^l\|_0\tau+\|\pi_N Q^{\frac{1}{2}}(W(t)-W(t_l))\|_0\right]\Bigg]1_l(t)\\
&\le C(T-t+1)^{\frac{1}{2}}e^{-\alpha(T-t)}\tau^{\frac{1}{2}},\\
E\|II_5^{l-1}\|_0 &\le e^{-\alpha(T-t)}(1-e^{-\alpha\tau})E\Bigg[\|u_N^{l-1}\|_1^2\|u_N^l\|_0\Bigg]\le Ce^{-\alpha(T-t)}\tau,
\end{aligned}
$$

and their integrals are also of order $\frac{1}{2}$. II_1^l, II_2^l and II_5^l can also be estimated in the same way, where we have used the fact that for any $T > 0$, the integral $\int_0^T (T-t+\tau)^{\frac{1}{2}}e^{-\alpha(T-t)}dt$ is bounded and $\sum_{l=1}^{M} 1_l(t) = 1$.

Other terms are proved in the same procedure by using the fact that

$$
\|b_l\|_{L^\infty}^2 \le C\|S(t-T)S_\tau^{M-l}\|^2_{\mathcal{L}(\dot{H}^1,\dot{H}^1)}[\|u_N^l\|_1^4+\|u_N^{l-1}\|_1^4+\|\pi_N Q^{\frac{1}{2}}\delta W_l\|_1^2]
$$

and $\|a_l b_l\|_{L^\infty} \le \frac{1}{2}[\|a_l\|_{L^\infty}^2+\|b_l\|_{L^\infty}^2]$. Finally, we have

$$
|II| \le C\tau^{\frac{1}{2}}. \tag{4.23}
$$

Next is the estimate of III, which is similar to the same part in the proof of Theorem 3.2.

$$\begin{aligned}
III &= \frac{1}{2}\sum_{l=1}^{M} E\int_0^T Tr\Big[\left(e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(e^{\alpha\tau}S_\tau^{M+1-l}\pi_N Q^{\frac{1}{2}}\right) \\
&\quad -\left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\Big]1_l(t)dt \\
&= \frac{1}{2}\sum_{l=1}^{M} E\int_0^T Tr\Big[\left(\left(e^{\alpha\tau}S_\tau^{M+1-l}-S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(\left(e^{\alpha\tau}S_\tau^{M+1-l}-S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right)\Big] \\
&\quad +2Tr\Big[\left(\left(e^{\alpha\tau}S_\tau^{M+1-l}-S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\Big]1_l(t)dt \\
&= \frac{1}{2}\sum_{l=1}^{M} E\int_0^T Tr\Big[e^{2\alpha\tau}\left(\left(S_\tau^{M+1-l}-S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(\left(S_\tau^{M+1-l}-S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right) \\
&\quad +2e^{2\alpha\tau}\left(\left(S_\tau^{M+1-l}-S(T-t)\right)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right) \\
&\quad +(e^{2\alpha\tau}-1)\left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)^* D^2 v_N \left(S(T-t)\pi_N Q^{\frac{1}{2}}\right)\Big]1_l(t)dt \\
&:= \frac{1}{2}\sum_{l=1}^{M} E\int_0^T (A_l+2B_l+C_l)1_l(t)dt,
\end{aligned}$$

where A_l, B_l and C_l satisfy

$$\begin{aligned}
E|A_l| &\le C\|S_\tau^{M+1-l}-S(T-t))\|^2_{\mathcal{L}(\dot{H}^2,L^2)}\|\pi_N Q^{\frac{1}{2}}\|^2_{\mathcal{L}(L^2,\dot{H}^2)}\|\phi\|_{C_b^2} \le C(T-t+\tau)e^{-2\alpha(T-t)}\tau, \\
E|B_l| &\le C\|S_\tau^{M+1-l}-S(T-t))\|_{\mathcal{L}(\dot{H}^2,L^2)}\|\pi_N Q^{\frac{1}{2}}\|^2_{\mathcal{L}(L^2,\dot{H}^2)}\|\phi\|_{C_b^2}\|S(T-t)\|_{\mathcal{L}(L^2,L^2)} \\
&\le C(T-t+\tau)^{\frac{1}{2}}e^{-2\alpha(T-t)}\tau^{\frac{1}{2}}
\end{aligned}$$

and

$$E|C_l| \le C\tau\|\pi_N Q^{\frac{1}{2}}\|^2_{\mathcal{L}(L^2,L^2)}\|\phi\|_{C_b^2}\|S(T-t)\|^2_{\mathcal{L}(L^2,L^2)} \le Ce^{-2\alpha(T-t)}\tau.$$

It follows

$$|III| \le C\tau^{\frac{1}{2}}. \tag{4.24}$$

We can conclude from Eqs. 4.21, 4.23 and 4.24 that,

$$\left|E\left[\phi(u_N(T))\right]-E\left[\phi(u_N^M)\right]\right| \le C\tau^{\frac{1}{2}},$$

where C is independent of T, M and N. □

Remark 5 For the linear case ($\lambda = 0$), as the weak convergence order depends heavily on the regularity of the solution, which depend only on the regularity of the initial value and noise, we can achieve higher order by increasing the regularity of the initial value and the noise. For example, the weak order turns out to be 1 if we assume $u_0 \in \dot{H}^4$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^4)} < \infty$. However, for the nonlinear case ($\lambda = \pm 1$), it is too technical to obtain the uniform higher regularity under proper assumptions, as a result, we work under the assumptions $u_0 \in \dot{H}^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$ and derive order $\frac{1}{2}$.

4.4 Convergence Order between Invariant Measures μ_N and μ_N^τ

Theorem 4.4 *For $\lambda = 0$ or -1, assume that $u_0 \in \dot{H}^2$ and $\|Q^{\frac{1}{2}}\|_{\mathcal{HS}(L^2,\dot{H}^2)} < \infty$, the error between invariant measures μ_N and μ_N^τ is of order $\frac{1}{2}$, i.e.,*

$$\left|\int_{V_N} \phi(y)d\mu_N(y) - \int_{V_N} \phi(y)d\mu_N^\tau(y)\right| < C\tau^{\frac{1}{2}}, \quad \forall\, \phi \in C_b^2(L^2).$$

Proof By the ergodicity of stochastic processes u_N and u_N^k, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E\phi\big(u_N(t)\big)dt = \int_{V_N} \phi(y)d\mu_N(y), \tag{4.25}$$

$$\lim_{M\to\infty} \frac{1}{M}\sum_{k=0}^{M-1} E\phi(u_N^k) = \int_{V_N} \phi(y)d\mu_N^\tau(y) \tag{4.26}$$

for any $\phi \in C_b^2(L^2)$. As the weak error is proved to be independent of step k and time t in Theorem 4.3, it turns out that for a fixed τ,

$$\begin{aligned}&\left|\int_{V_N} \phi(y)d\mu_N(y) - \int_{V_N} \phi(y)d\mu_N^\tau(y)\right| \\ \le& \lim_{\substack{M\to\infty,\\ T=M\tau\to\infty}} \frac{1}{T}\sum_{k=0}^{M-1}\int_{t_k}^{t_{k+1}} \left|E\phi\big(u_N(t)\big) - E\phi(u_N^k)\right| dt \le C\tau^{\frac{1}{2}}.\end{aligned}$$

□

Remark 6 For the case $\lambda = 1$, if the 1-norm and 2-norm of u_N^k is also uniformly bounded, we can also get order $\frac{1}{2}$ for both time-independent weak error and error between invariant measures. If not, based on the fact $\|\cdot\|_{s+1} \le N\|\cdot\|_s$, we can get the weak error depend on N

$$\left|E[\phi(u_N(T))] - E[\phi(u_N^M)]\right| \le CN^4\tau^{\frac{1}{2}},$$

as well as the error between invariant measures.

5 Numerical Experiments

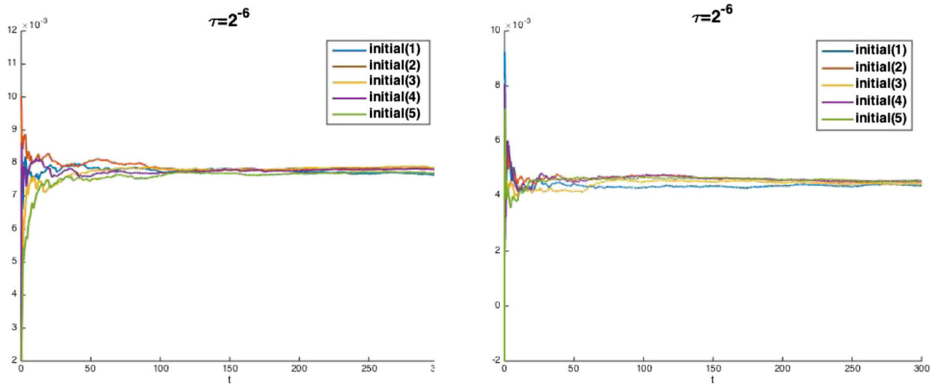
This section provides numerical experiments to test the longtime behavior of scheme (4.1) for the case $\lambda = 0$. Based on the spatial semi-discretization in stochastic ordinary differential equation form Eq. 3.2

$$da_m(t) = -\mathbf{i}(m\pi)^2 a_m(t)dt - \alpha a_m(t)dt + \sqrt{\eta_m}d\beta_m(t), \quad 1 \le m \le N,$$

we derive an equivalent form of the full discretization (4.1) as

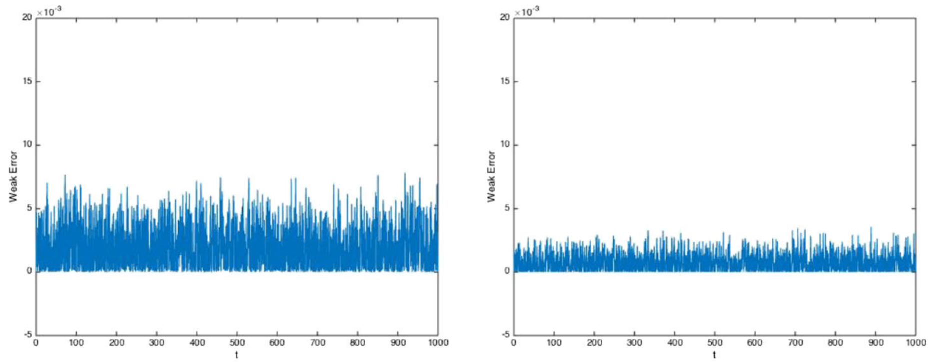
$$\vec{a}^k - e^{-\alpha\tau}\vec{a}^{k-1} = -\mathbf{i}\tau\pi^2 \begin{pmatrix} 1 & & \\ & \ddots & \\ & & N^2 \end{pmatrix} \vec{a}^k + \begin{pmatrix} \sqrt{\eta_1}\delta_k\beta_1 \\ \vdots \\ \sqrt{\eta_N}\delta_k\beta_N \end{pmatrix},$$

where $\vec{a}^k := (a_1^k, \cdots, a_N^k)^T$ is an approximation of $\vec{a}(t) := (a_1(t), \cdots, a_N(t))^T$ and $\delta_k\beta_m = \beta_m(t_k) - \beta_m(t_{k-1})$ for $1 \le m \le N$. In the sequel, we take $\alpha = 1$, $N = 100$.

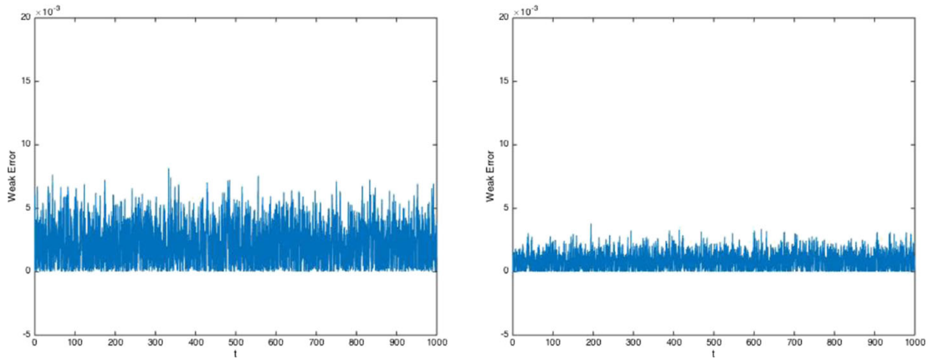


(a) $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2)$ (b) $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2})$

Fig. 1 The temporal averages $\frac{1}{M+1}\sum_{k=0}^{M} E[\phi(\vec{a}^k)]$ started from different initial values ($\tau = 2^{-6}$, $T = 300$)

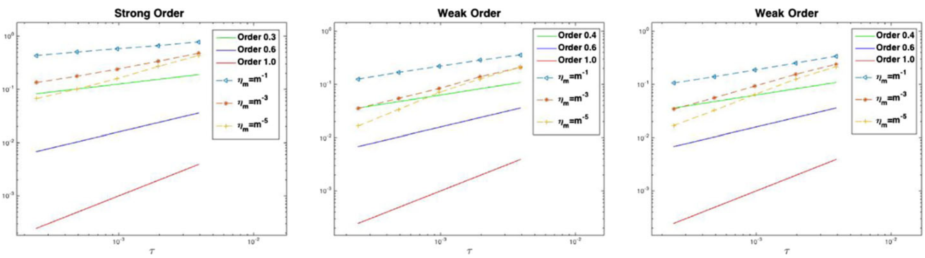


(a) $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2), \tau = 2^{-4}$ (b) $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2), \tau = 2^{-6}$



(c) $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2}), \tau = 2^{-4}$ (d) $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2}), \tau = 2^{-6}$

Fig. 2 The weak error $E[\phi(\vec{a}(t_k)) - \phi(\vec{a}^k)]$ for different ϕ and step size τ with $t_k = k\tau \in [0, T]$ and $T = 10^3$



(a) Strong Order (b) $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2)$ (c) $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2})$

Fig. 3 The strong and weak orders for noise in L^2, $\dot{H}^2$ and $\dot{H}^4$, i.e., $\eta_m = m^{-1}, m^{-3}, m^{-5}$. ($T = \frac{1}{2}$, $\tau \in \{2^{-i}, 5 \le i \le 9\}$)

In Fig. 1, the temporal averages $\frac{1}{M+1}\sum_{k=1}^{M} E[\phi(\vec{a}^k)]$ of the fully discrete scheme started from five different initial values $\text{initial}(1) = (1, 0, \cdots, 0)^T$, $\text{initial}(2) = (0.0003\mathbf{i}, 0, \cdots, 0)^T$, $\text{initial}(3) = \left(\sin\left(\frac{1}{101}\pi\right), \sin\left(\frac{2}{101}\pi\right), \cdots, \sin\left(\frac{100}{101}\pi\right)\right)^T$, $\text{initial}(4) = \left(\frac{2+\mathbf{i}}{20}\right)(1, 2, \cdots, 100)^T$ and $\text{initial}(5) = \left(\exp\left(-\frac{\mathbf{i}}{50}\right), \exp\left(-\frac{2\mathbf{i}}{50}\right), \cdots, \exp\left(-\frac{100\mathbf{i}}{50}\right)\right)^T$ will converge to the same value with error $\tau^{\frac{1}{2}}$ before time T, where $\tau = 2^{-6}$ and $T = 300$. This result verifies the ergodicity of the numerical solution: the temporal averages converge to the spatial average, which is a constant, for almost every initial values in the whole space. We choose 500 realizations to approximate the expectations in Figs. 1 and 2, and choose 1000 realizations in Fig. 3.

In Figs. 2 and 3, we fix the initial value $u_0(x)$ as $\sqrt{2}\sin(\pi x)$, such that $a_m(0) = (u_0, e_m)$ and $\vec{a}^0 = \vec{a}(0) = (1, 0, \cdots, 0)^T$. Figure 2 displays the weak error $E[\phi(\vec{a}(t_k)) - \phi(\vec{a}^k)]$ over long time $T = 10^3$ for different time step sizes and test functions: (a) $\tau = 2^{-4}$, $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2)$ (b) $\tau = 2^{-6}$, $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2)$, (c) $\tau = 2^{-4}$, $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2})$ and (d) $\tau = 2^{-6}$, $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2})$. The reference values are generated for the time step size $\tau = 2^{-8}$, and the noise is chosen in $\dot{H}^2$, i.e., $\eta_m = m^{-3}$. Figure 2 shows that the weak error is independent of time interval and can be controlled by $C\tau^{\frac{1}{2}}$, which coincides with our theoretical results. Figure 3 displays both (a) the strong convergence order and the rates of weak convergence for (b) $\phi(\vec{a}) = \exp(-\|\vec{a}\|_{l^2}^2)$ or (c) $\phi(\vec{a}) = \sin(\|\vec{a}\|_{l^2})$. The reference values are generated for the time step size $\tau = 2^{-14}$. As the initial value $u_0(x) = \sqrt{2}\sin(\pi x)$ is regular enough, both the strong and weak convergence order depend heavily on the regularity of the noise for the linear case. It shows in Fig. 3 that the orders slightly increase as the noise from L^2 via $\dot{H}^2$ to $\dot{H}^4$ (i.e., η_m from m^{-1} via m^{-3} to m^{-5}), which verifies Remark 5. Noticing that the orders are a little bit better than the theoretical results, because the truncation of the noise makes the noise more regular than it should be, which increases the orders slightly. Numerical tests also shows that the weak convergence order is almost the same as the strong convergence order, which is similar to the statement in [7] (Remark 5.11).

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Appendix

The Proof of Proposition 3.1

i) As it is proved in Part 3 of Theorem 3.1 that $E\|u_N(t)\|_0^2 < C$, we assume further that $E\|u_N(t)\|_0^{2n} < C,\ \forall\ n = 1, \cdots, p-1$. Denoting $dM_1 := 2Re\left(u_N, \pi_N Q^{\frac{1}{2}} dW\right)$, then Itô's formula and Eq. 3.5 yields

$$
\begin{aligned}
d\|u_N(t)\|_0^{2p} &= p\|u_N(t)\|_0^{2(p-1)} d\|u_N(t)\|_0^2 + \frac{1}{2}p(p-1)\|u_N(t)\|_0^{2(p-2)} d\langle M_1\rangle \\
&\le -2\alpha p\|u_N(t)\|_0^{2p} dt + p\|u_N(t)\|_0^{2(p-1)} dM_1(t) \\
&\quad +2p(2p-1)\sum_{m=1}^{N} \eta_m \|u_N(t)\|_0^{2(p-1)} dt,
\end{aligned}
$$

where $\langle\cdot\rangle$ denotes the quadratic variation process and in the last step we used the fact

$$
\begin{aligned}
d\langle M_1\rangle &= 4\left\langle Re\sum_{m=1}^{N}\int_0^1 \overline{u}_N(s)\sqrt{\eta_m} e_m(x)dx(d\beta_{m,1} + \mathbf{i}d\beta_{m,2})\right\rangle \\
&= 4\sum_{m=1}^{N}\left[\left(Re\int_0^1 \overline{u}_N(t,x)\sqrt{\eta_m} e_m(x)dx\right)^2 + \left(Im\int_0^1 \overline{u}_N(t,x)\sqrt{\eta_m} e_m(x)dx\right)^2\right] dt \\
&\le 8\sum_{m=1}^{N} \eta_m \|u_N(t)\|_0^2 dt.
\end{aligned}
$$

Taking expectation on both sides of above equation, we obtain

$$
\begin{aligned}
\frac{d}{dt}E\|u_N(t)\|_0^{2p} &\le -2\alpha p E\|u_N(t)\|_0^{2p} + 2p(2p-1)\sum_{m=1}^{N}\eta_m E\|u_N(t)\|_0^{2(p-1)} \\
&\le -2\alpha p E\|u_N(t)\|_0^{2p} + C
\end{aligned}
$$

by induction. Then multiplying $e^{2\alpha pt}$ to both sides of above equation yields the result.

ii) The proof in this part is similar to the proof of Lemma 2.5 in [8]. According to the Gagliardo-Nirenberg interpolation inequality, there exists a positive constant c_0, such that

$$
\frac{5}{8}\lambda\|u_N(t)\|_{L^4}^4 \le \|u_N(t)\|_{L^4}^4 \le \frac{1}{4}\|\nabla u_N(t)\|_0^2 + \frac{1}{2}c_0\|u_N(t)\|_0^6. \tag{1}
$$

Thus,

$$
\begin{aligned}
0 \le \mathcal{H}(u_N(t)) &:= \frac{1}{2}\|\nabla u_N(t)\|_0^2 - \frac{\lambda}{4}\|u_N(t)\|_{L^4}^4 + c_0\|u_N(t)\|_0^6 \\
&\le \frac{2}{3}\left(\|\nabla u_N(t)\|_0^2 - \lambda\|u_N(t)\|_{L^4}^4 + 2c_0\|u_N(t)\|_0^6\right).
\end{aligned} \tag{2}
$$

Applying Itô's formula to $\mathcal{H}(u_N(t))$, it leads to

$$\begin{aligned} d\mathcal{H}(u_N(t)) = & \Bigg[-\alpha\|\nabla u_N(t)\|_0^2 + \alpha\lambda\|u_N(t)\|_{L^4}^4 - 6\alpha c_0\|u_N(t)\|_0^6 \\ & -2\lambda \int_0^1 |u_N|^2 \sum_{m=1}^N \eta_m |e_m|^2 dx \\ & + \sum_{m=1}^N m^2 \eta_m + 6c_0\|u_N(t)\|_0^4 \sum_{m=1}^N \eta_m \\ & +12c_0\|u_N(t)\|_0^2 \|\pi_N Q^{\frac{1}{2}} u_N(t)\|_0^2 \Bigg] dt \\ & +6c_0\|u_N(t)\|_0^4 Re\left(u_N, \pi_N Q^{\frac{1}{2}} dW\right) \\ & -Re\left(\Delta u_N(t) + \lambda|u_N(t)|^2 u_N(t), \pi_N Q^{\frac{1}{2}} dW\right), \end{aligned}$$

where we have used the fact $((Id - \pi_N)v, v_N) = 0,\ \forall\, v \in \dot{H}^0,\ v_N \in V_N$. By the following estimates

$$-2\lambda \int_0^1 |u_N|^2 \sum_{m=1}^N \eta_m |e_m|^2 dx \le 0,$$

$$6c_0\|u_N(t)\|_0^4 \sum_{m=1}^N \eta_m + 12c_0\|u_N(t)\|_0^2\|\pi_N Q^{\frac{1}{2}} u_N(t)\|_0^2 \le 4\alpha c_0\|u_N(t)\|_0^6 + C$$

and Eq. 1, we have

$$\begin{aligned} d\mathcal{H}(u_N(t)) \le & \Bigg[-\alpha\|\nabla u_N(t)\|_0^2 + \alpha\lambda\|u_N(t)\|_{L^4}^4 & (3) \\ & -2\alpha c_0\|u_N(t)\|_0^6 + \sum_{m=1}^N m^2\eta_m + C\Bigg] dt \\ & +6c_0\|u_N(t)\|_0^4 Re\left(u_N(t), \pi_N Q^{\frac{1}{2}} dW(t)\right) & (4) \\ & -Re\left(\Delta u_N(t) + \lambda|u_N(t)|^2 u_N(t), \pi_N Q^{\frac{1}{2}} dW\right) \\ \le & -\frac{3}{2}\alpha\mathcal{H}(u_N(t))dt + Cdt + dM_2, & (5) \end{aligned}$$

where

$$dM_2 := 6c_0\|u_N\|_0^4 Re\left(u_N, \pi_N Q^{\frac{1}{2}} dW\right) - Re\left(\Delta u_N + \lambda|u_N|^2 u_N, \pi_N Q^{\frac{1}{2}} dW\right).$$

Taking expectation, we derive

$$dE\mathcal{H}(u_N(t)) \le -\frac{3}{2}\alpha E\mathcal{H}(u_N(t))dt + Cdt.$$

Hence, by multiplying $e^{\frac{3}{2}\alpha t}$ to both sides of the equation above and then taking integral from 0 to t, we get the uniform boundedness for $p = 1$. By induction, we assume that the results hold for $p-1$. Then, based on the following estimates (see [8])

$$\left\langle 6\|u_N\|_0^4 Re\left(u_N, \pi_N Q^{\frac{1}{2}} dW\right)\right\rangle^2 \leq C\|Q^{\frac{1}{2}}\|^2_{\mathcal{HS}(L^2,L^2)}\|u_N\|_0^{10}dt,$$

$$\left\langle Re\left(\Delta u_N + \lambda|u_N|^2u_N, \pi_N Q^{\frac{1}{2}} dW\right)\right\rangle^2 \leq C\|Q^{\frac{1}{2}}\|^2_{\mathcal{HS}(L^2,\dot{H}^1)}\left(\|\nabla u_N\|_0^2 + \|u_N\|_0^{10}\right)dt$$

and Eq. 5, we have

$$\begin{aligned} d\mathcal{H}(u_N(t))^p &= p\mathcal{H}(u_N(t))^{p-1}d\mathcal{H}(u_N(t)) + \frac{1}{2}p(p-1)\mathcal{H}(u_N(t))^{p-2}d\langle M_2\rangle \\ &\leq -\frac{3}{2}\alpha p\mathcal{H}(u_N(t))^p dt + Cp\mathcal{H}(u_N(t))^{p-1}dt + p\mathcal{H}(u_N(t))^{p-1}dM_2 \\ &\quad + Cp(p-1)\mathcal{H}(u_N(t))^{p-2}\left(\|\nabla u_N(t)\|_0^2 + \|u_N(t)\|_0^{10}\right)dt. \end{aligned} \tag{6}$$

From Eq. 1, we deduce that

$$\mathcal{H}(u_N(t)) \geq \begin{cases} \frac{1}{2}\|\nabla u_N(t)\|_0^2 + c_0\|u_N(t)\|_0^6, & \lambda = 0 \text{ or } -1, \\ \frac{7}{16}\|\nabla u_N(t)\|_0^2 + \frac{7}{8}c_0\|u_N(t)\|_0^6, & \lambda = 1. \end{cases}$$

As a result, the last term in Eq. 6 can be estimated as

$$\begin{aligned} &Cp(p-1)\mathcal{H}(u_N(t))^{p-2}\left(\|\nabla u_N(t)\|_0^2 + \|u_N(t)\|_0^{10}\right) \\ &\leq \left(C\mathcal{H}(u_N(t)) + C\mathcal{H}(u_N(t))^{\frac{5}{3}}\right)\mathcal{H}(u_N(t))^{p-2} \leq C\mathcal{H}(u_N(t))^{p-1} + \frac{1}{2}\alpha p\mathcal{H}(u_N(t))^p, \end{aligned} \tag{7}$$

where in the last step we used the inequality of arithmetic and geometric means

$$C(\mathcal{H}(u_N(t))^2\cdot\mathcal{H}(u_N(t))^2\cdot\mathcal{H}(u_N(t)))^{\frac{1}{3}} \leq \frac{\frac{3}{4}\alpha p\mathcal{H}(u_N(t))^2 + \frac{3}{4}\alpha p\mathcal{H}(u_N(t))^2 + C\mathcal{H}(u_N(t))}{3}.$$

Gethering Eqs. 6 and 7 and taking expectation, we obtain

$$dE\mathcal{H}(u_N(t))^p \leq -\alpha pE\mathcal{H}(u_N(t))^p dt + Cdt$$

by induction, which complete the proof by multiplying $e^{\alpha pt}$ on both sides of above equation.

iii) We define a functional

$$f(u) = \int_0^1 |\Delta u|^2 dx + \lambda Re\int_0^1 (\Delta\overline{u})|u|^2 u dx,$$

which satisfies

$$\|\Delta u\|_0^2 \leq 2f(u) + C\|u\|_1^6 \tag{8}$$

based on the continuous embedding $H^1 \hookrightarrow L^6$ and $\left|\lambda Re\int_0^1 \Delta\overline{u}|u|^2 u dx\right| \leq \frac{1}{2}\|\Delta u\|_0^2 + \frac{1}{2}\|u\|_{L^6}^6 \leq \frac{1}{2}\|\Delta u\|_0^2 + C\|u\|_1^6$. The Itô's formula applied to $f(u_N)$ yields

$$\begin{aligned} df(u_N) &= Df(u_N)\left(\left(\mathbf{i}\Delta u_N + \mathbf{i}\lambda|u_N|^2u_N - \alpha u_N\right)dt\right) + Df(u_N)\left(\pi_N Q^{\frac{1}{2}}dW\right) \\ &\quad + \frac{1}{2}D^2 f(u_N)(\pi_N Q^{\frac{1}{2}}dW, \pi_N Q^{\frac{1}{2}}dW) \\ &=: \mathcal{A} + \mathcal{B} + \mathcal{C}, \end{aligned} \tag{9}$$

where

$$
\begin{aligned}
Df(u)(\varphi) &= Re\int_0^1 \Big[2\Delta\overline{u}\Delta\varphi + 2\lambda(\Delta\overline{u})uRe(\overline{u}\varphi) + \lambda(\Delta\overline{u})|u|^2\varphi \\
&\quad +\lambda(\Delta(|u|^2u))\overline{\varphi}\Big]dx, \\
D^2 f(u)(\varphi,\psi) &= Re\int_0^1 \Big[2\Delta\overline{\varphi}\Delta\psi + 2\lambda(\Delta\overline{u})uRe(\overline{\varphi}\psi) + 2\lambda(\Delta\overline{u})\varphi Re(\overline{u}\psi) \\
&\quad +2\lambda(\Delta\overline{\varphi})uRe(\overline{u}\psi) \\
&\quad +2\lambda(\Delta\overline{u})\psi Re(\overline{\varphi}u) + 2\lambda(\Delta\overline{\psi})uRe(\overline{u}\varphi) + \lambda(\Delta\overline{\varphi})|u|^2\psi \\
&\quad +\lambda(\Delta\overline{\psi})|u|^2\varphi\Big]dx
\end{aligned}
$$

and $E[\mathcal{B}] = 0$. Now we estimate $\mathcal{A}$ and $\mathcal{C}$ respectively.

$$
\begin{aligned}
E[\mathcal{A}] &= -2\alpha E[f(u_N)]dt + ReE\int_0^1 \Big[4\lambda\mathbf{i}(\Delta\overline{u}_N)u_N|\nabla u_N|^2 \\
&\quad +2\lambda\mathbf{i}(\Delta\overline{u}_N)\overline{u}_N(\nabla u_N)^2\Big]dxdt \\
&\quad +ReE\int_0^1 \Big[\lambda^2\mathbf{i}(\Delta\overline{u}_N)|u_N|^4 - 4\alpha\lambda(\Delta\overline{u}_N)u_N|u_N|^2\Big]dxdt \\
&\quad +ReE\int_0^1 \Big[-4\alpha\lambda|u_N|^2|\nabla u_N|^2 - 2\alpha\lambda(\nabla u_N)^2\overline{u}_N^2\Big]dxdt \\
&=: -2\alpha E[f(u_N)]dt + \mathcal{A}_1 dt + \mathcal{A}_2 dt + \mathcal{A}_3 dt,
\end{aligned}
$$

where we have used the fact $\Delta(|u|^2u) = 2|u|^2\Delta u + 4u|\nabla u|^2 + 2\overline{u}(\nabla u)^2 + u^2\Delta\overline{u}$ and $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are estimated as follows.

$$
\begin{aligned}
|\mathcal{A}_1| &:= \left|ReE\int_0^1 \Big[4\lambda\mathbf{i}(\Delta\overline{u}_N)u_N|\nabla u_N|^2 + 2\lambda\mathbf{i}(\Delta\overline{u}_N)\overline{u}_N(\nabla u_N)^2\Big]dx\right| \\
&\le \frac{\alpha}{16}E\|\Delta u_N\|_0^2 + CE\Big[\|u_N\|_{L^\infty}^2\|\nabla u_N\|_{L^4}^2\Big] \\
&\le \frac{\alpha}{16}E\|\Delta u_N\|_0^2 + CE\Big[\|u_N\|_{L^\infty}^4 + \|\Delta u_N\|_0\|\nabla u_N\|_0^3\Big] \\
&\le \frac{\alpha}{8}E\|\Delta u_N\|_0^2 + CE\Big[\|u_N\|_1^4 + \|u_N\|_1^6\Big] \\
&\le \frac{\alpha}{8}E\|\Delta u_N\|_0^2 + C,
\end{aligned}
$$

where we have used the uniform boundedness of $\|u_N\|_1^{2p}$ for $p \ge 1$ in ii), the continuous embedding $H^1 \hookrightarrow L^\infty$ for $\mathbb{R}^1$ and the interpolation of L^4 between L^2 and H^1. Similarly, based on the continuous embedding $H^1 \hookrightarrow L^6$ and $H^1 \hookrightarrow L^8$, we have

$$
\begin{aligned}
|\mathcal{A}_2| &:= \left|ReE\int_0^1 \Big[\lambda^2\mathbf{i}(\Delta\overline{u}_N)|u_N|^4 - 4\alpha\lambda(\Delta\overline{u}_N)u_N|u_N|^2\Big]dx\right| \\
&\le \frac{\alpha}{8}E\|\Delta u_N\|_0^2 + CE[\|u_N\|_{L^8}^8 + \|u_N\|_{L^6}^6] \le \frac{\alpha}{8}E\|\Delta u_N\|_0^2 + C
\end{aligned}
$$

and

$$
|\mathcal{A}_3| := \left|ReE\int_0^1 \Big[-4\alpha\lambda|u_N|^2|\nabla u_N|^2 - 2\alpha\lambda(\nabla u_N)^2\overline{u}_N^2\Big]dx\right| \le CE\|u_N\|_1^4 \le C.
$$

Thus, we obtain

$$E[\mathcal{A}] \leq -2\alpha E[f(u_N)]dt + \frac{\alpha}{4}E\|\Delta u_N\|_0^2 + C.$$

The estimate of $\mathcal{C}$ is similar with that of $\mathcal{A}$, and we derive $E[\mathcal{C}] \leq \frac{\alpha}{4}E\|\Delta u_N\|_0^2 + C$. Taking expectation on both sides of Eq. 9 yields

$$dEf(u_N) + 2\alpha Ef(u_N)dt \leq \frac{\alpha}{2}E\|\Delta u_N\|_0^2 dt + Cdt \leq \alpha Ef(u_N)dt + Cdt.$$

Multiplying both sides of above equation by $e^{\alpha t}$ and taking integral from 0 to t, we conclude the uniform boundedness of $Ef(u_N(t))$

$$Ef(u_N(t)) \leq e^{-\alpha t}Ef(u_N(0)) + \frac{C}{\alpha}(1 - e^{-\alpha t}),$$

which yields the uniform boundedness of $E\|\Delta u_N\|_0^2$ based on Eq. 8. As the norm $\|u_N\|_2$ is equivalent to $\|\Delta u_N\|_0$ under Dirichlet boundary condition, we complete the proof.

□

The Proof of Uniqueness of the Solution for Eq. 4.1

Suppose that U and W are two solutions of the scheme, then it follows

$$U - W = \mathbf{i}\tau\Delta\big(U - W\big) + \mathbf{i}\lambda\frac{\tau}{2}\pi_N\Big[\big(|U|^2U - |W|^2W\big) + |e^{-\alpha\tau}u_N^{k-1}|^2(U - W)\Big].$$

Multiply the equation above by $\overline{U} - \overline{W}$, integrate in space and take the real and imaginary part respectively, we have

$$\|U - W\|_0^2 \leq \frac{\tau}{2}\|f(U) - f(W)\|_{L^{\frac{4}{3}}}\|U - W\|_{L^4},$$
$$\|\nabla(U - W)\|_0^2 \leq \frac{1}{2}\|f(U) - f(W)\|_{L^{\frac{4}{3}}}\|U - W\|_{L^4} + \frac{\lambda}{2}\|e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2\|U - W\|_{L^4}^2,$$

where $f(U) := |U|^2U$ and

$$\|f(U) - f(W)\|_{L^{\frac{4}{3}}}$$
$$= \left(\int_0^1 \Big||U|^2U - |W|^2W\Big|^{\frac{4}{3}}dx\right)^{\frac{3}{4}} = \left(\int_0^1 \Big||U|^2(U - W) + |W|^2(U - W) + UW(\overline{U} - \overline{W})\Big|^{\frac{4}{3}}dx\right)^{\frac{3}{4}}$$
$$\leq \left(\int_0^1 \Big||U|^2 + |W|^2 + |UW|\Big|^2dx\right)^{\frac{1}{2}}\left(\int_0^1 |U - W|^4dx\right)^{\frac{1}{4}} \leq \big\||U| + |W|\big\|_{L^4}^2\|U - W\|_{L^4}.$$

Since

$$\begin{aligned}\|U-W\|_{L^4}^4 &\le \|U-W\|_0^3\|\nabla(U-W)\|_0\\ &\le \left(\frac{\tau}{2}\|f(U)-f(W)\|_{L^{\frac{4}{3}}}\|U-W\|_{L^4}\right)^{\frac{3}{2}}\left(\frac{1}{2}\|f(U)-f(W)\|_{L^{\frac{4}{3}}}\|U-W\|_{L^4}\right.\\ &\qquad\left.+\frac{|\lambda|}{2}\|e^{-\alpha\tau}u_N^{k-1}\|_{L^4}^2\|U-W\|_{L^4}^2\right)^{\frac{1}{2}}\\ &\le \frac{1}{4}\tau^{\frac{3}{2}}\big\||U|+|W|\big\|_{L^4}^3\left(\big\||U|+|W|\big\|_{L^4}^2+|\lambda|\|u_N^{k-1}\|_{L^4}^2\right)^{\frac{1}{2}}\|U-W\|_{L^4}^4\\ &\le \frac{1}{4}\tau^{\frac{3}{2}}\left(\big\||U|+|W|\big\|_{L^4}^4+|\lambda|\big\||U|+|W|\big\|_{L^4}^3\|u_N^{k-1}\|_{L^4}\right)\|U-W\|_{L^4}^4,\end{aligned}$$

if $U \neq W$, then

$$\begin{aligned}1 &\le \frac{1}{4}\tau^{\frac{3}{2}}\left(\big\||U|+|W|\big\|_{L^4}^4+|\lambda|\big\||U|+|W|\big\|_{L^4}^3\|u_N^{k-1}\|_{L^4}\right)\\ &\le C_0\tau^{\frac{3}{2}}\left(\big\||U|+|W|\big\|_{L^4}^4+|\lambda|\big\||U|+|W|\big\|_{L^4}^6+|\lambda|\|u_N^{k-1}\|_{L^4}^2\right).\end{aligned}$$

For cases $\lambda = 0$ or -1, the L^4-norm of the solutions are uniformly bounded. So $C_0\tau^{\frac{3}{2}} > 1$, which do not hold when τ is sufficiently small. For case $\lambda = 1$, according to the fact that

$$\big\||U|+|W|\big\|_{L^4}^6 \le \big\||U|+|W|\big\|_0^{\frac{3}{2}}\big\|\nabla(|U|+|W|)\big\|_0^{\frac{9}{2}} \le N^{\frac{9}{2}}\big\||U|+|W|\big\|_0^6,$$

we have $C_0N^{\frac{9}{2}}\tau^{\frac{3}{2}} > 1$, which is also a contradiction when τ is sufficiently small.

Thus, the numerical solution for Eq. 4.1 is unique. □

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